Fixpoints

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Motivation
The previous presentation provided us the "natural meaning" of positive definite rules:

Theorem (Minimal Herbrand Model of a Definite Program)

- Each set $S$ of positive definite rules has a unique minimal Herbrand model.
- This model is the intersection of all Herbrand models of $S$.
- It satisfies precisely those ground atoms that are logical consequences of $S$.

The theorem states what the unique minimal Herbrand model is, but not how to calculate it.
How to evaluate sets of clauses?

Example (Set of universal definite rules)

\[
\begin{align*}
\text{feeds}_\text{milk}(\text{betty}) & \leftarrow T \\
\text{lays}_\text{eggs}(\text{betty}) & \leftarrow T \\
\text{has}_\text{spines}(\text{betty}) & \leftarrow T \\
\text{monotreme}(X) & \leftarrow \text{lays}_\text{eggs}(X), \text{feeds}_\text{milk}(X) \\
\text{echidna}(X) & \leftarrow \text{monotreme}(X), \text{has}_\text{spine}(X) \\
\text{blub}(X) & \leftarrow \text{bla}(X), \text{echidna}(X)
\end{align*}
\]

Intuitive approach:

1. start with no known facts: $\emptyset$
2. evaluate all antecedents with given set of facts
3. add consequents to set of facts if antecedent evaluated to true in step 2
4. repeat steps 2 and 3 until there are no more changes
How to evaluate sets of clauses?

Example

\[
\begin{align*}
\text{feeds\_milk}(\text{betty}) & \leftarrow T \\
\text{lays\_eggs}(\text{betty}) & \leftarrow T \\
\text{has\_spines}(\text{betty}) & \leftarrow T \\
\text{monotreme}(X) & \leftarrow \text{lays\_eggs}(X), \text{feeds\_milk}(X) \\
\text{echidna}(X) & \leftarrow \text{monotreme}(X), \text{has\_spine}(X) \\
\text{blub}(X) & \leftarrow \text{bla}(X), \text{echidna}(X)
\end{align*}
\]

\[
\begin{align*}
F_0 &= \{\} \\
F_1 &= \{\text{feeds\_milk}(\text{betty}), \text{lays\_eggs}(\text{betty}), \text{has\_spines}(\text{betty})\} \\
F_2 &= \{\text{feeds\_milk}(\text{betty}), \text{lays\_eggs}(\text{betty}), \text{has\_spines}(\text{betty}), \text{monotreme}(\text{betty})\} \\
F_3 &= \{\text{feeds\_milk}(\text{betty}), \text{lays\_eggs}(\text{betty}), \text{has\_spines}(\text{betty}), \text{monotreme}(\text{betty}), \text{echidna}(\text{betty})\} \\
F_4 &= \{\text{feeds\_milk}(\text{betty}), \text{lays\_eggs}(\text{betty}), \text{has\_spines}(\text{betty}), \text{monotreme}(\text{betty}), \text{echidna}(\text{betty})\}
\]
Questions

1. Does the approach terminate?
2. Is there always a solution?
3. Is the solution unique?
4. What does the solution represent?
Roadmap

1. Fixpoint Theory
   1. Theorem of Knaster and Tarski
   2. Theorem of Kleene

2. Application to Logic Programming
   1. Immediate Consequence Operator
   2. The unique Herbrand Model as a fixpoint
Fixpoints
Operators

Definition (Operator)

Let $X$ be a set. Then an **operator** is a mapping $\Gamma : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$.

Example (Operator)

Let $X = \{1, 2\}$. Then $\Gamma_1$ is an operator on $X$:

\[
\begin{align*}
\Gamma_1 (\emptyset) &= \{1, 2\} \\
\Gamma_1 (\{1\}) &= \{2\} \\
\Gamma_1 (\{2\}) &= \emptyset \\
\Gamma_1 (\{1, 2\}) &= \{1\}
\end{align*}
\]
Monotonic Operators: Preserve Subset Relationship

Definition (Monotonic Operator)

Let $\Gamma$ be an operator on $X$. Then $\Gamma$ is **monotonic** iff. for all $M, M' \subseteq X$

$$M \subseteq M' \Rightarrow \Gamma(M) \subseteq \Gamma(M')$$

Example (Monotonic Operators)

Let $X = \{1, 2\}$. Then $\Gamma_2$ is a monotonic operator on $X$:

$$\Gamma_2(\emptyset) = \{1\} \quad \Gamma_2(\{1\}) = \{1\} \quad \Gamma_2(\{2\}) = \{1\} \quad \Gamma_2(\{1, 2\}) = \{1, 2\}$$

Check:

$$\Gamma_2(\emptyset) \subseteq \Gamma_2(\{1\}) \quad \Gamma_2(\emptyset) \subseteq \Gamma_2(\{2\}) \quad \Gamma_2(\emptyset) \subseteq \Gamma_2(\{1, 2\})$$

$$\Gamma_2(\{1\}) \subseteq \Gamma_2(\{1, 2\}) \quad \Gamma_2(\{2\}) \subseteq \Gamma_2(\{1, 2\})$$

$$\Gamma_2(M') \subseteq \Gamma_2(M') \quad \text{for } M' \subseteq X$$
Directed Sets

Definition (Directed Set)

Let $X \neq \emptyset$ be a set. Then a set $Y \subseteq \mathcal{P}(X)$ is directed if for every finite subset $Y_{\text{fin}} \subseteq Y$ there is $M \in Y$ such that

$$\bigcup Y_{\text{fin}} \subseteq M$$

Example (Directed Set)

Let $X = \{1, 2\}$. Then $Y_{14} = \{\{1\}, \{2\}\}$, $Y_{15} = \{\emptyset, \{1\}, \{2\}\}$ and $Y_{16} = \emptyset$ are not directed. The directed sets regarding $X$:

- $Y_1 = \{\emptyset\}$
- $Y_4 = \{\{1, 2\}\}$
- $Y_7 = \{\{1\}, \{2\}, \{1, 2\}\}$
- $Y_2 = \{\{1\}\}$
- $Y_5 = \{\{1\}, \{1, 2\}\}$
- $Y_6 = \{\{2\}, \{1, 2\}\}$
- $Y_8 + k = \{\emptyset\} \cup Y_{2+k}$ for $k = 0, \ldots, 5$
Continuous Operators: Preserve Directed Suprema

Definition (Continuous Operator)

Let $\Gamma$ be an operator on $X \neq \emptyset$. Then $\Gamma$ is continuous iff. for each directed set $Y \subseteq \mathcal{P}(X)$ holds

$$\Gamma \left( \bigcup Y \right) = \bigcup \Gamma [Y] = \bigcup \{\Gamma(Y') \mid Y' \in Y\}$$

Example (Continuous Operator)

Let $X = \{1, 2\}$. Then $\Gamma_2$ is a continuous operator on $X$:

$$\Gamma_2(\emptyset) = \{1\} \quad \Gamma_2(\{1\}) = \{1\} \quad \Gamma_2(\{2\}) = \{1\} \quad \Gamma_2(\{1, 2\}) = \{1, 2\}$$

Check (e.g. $Y_1 = \emptyset$, $Y_5 = \{\{1\}, \{1, 2\}\}$ and $Y_{13} = \emptyset, \{1\}, \{2\}, \{1, 2\}$):

$$\Gamma_2(\emptyset) = \Gamma_2(\emptyset)$$
$$\Gamma_2(\{1\} \cup \{1, 2\}) = \Gamma_2(\{1\}) \cup \Gamma_2(\{1, 2\})$$
$$\Gamma_2(\emptyset \cup \{1\} \cup \{2\} \cup \{1, 2\}) = \Gamma_2(\emptyset) \cup \Gamma_2(\{1\}) \cup \Gamma_2(\{2\}) \cup \Gamma_2(\{1, 2\})$$
Lemma (Continuous implies Monotonic)

Let $\Gamma$ be a continuous operator on $X \neq \emptyset$. Then $\Gamma$ is monotonic.

Example (Continuous Operator is Monotonic)

Let $X = \{1, 2\}$. $\Gamma_2$ is both a continuous and monotonic operator on $X$.

Example

The inverse implication does not hold in general!
Fixpoints

Definition (Fixpoint)

Let \( \Gamma \) be an operator on \( X \). Then \( X' \subseteq X \) is a **fixpoint** iff.

\[
\Gamma(X') = X'
\]

Example (Fixpoint)

Let \( X = \{1, 2\} \). Given the operator \( \Gamma_2 \) on \( X \)

\[
\begin{align*}
\Gamma_2(\emptyset) &= \{1\} \\
\Gamma_2(\{1\}) &= \{1\} \\
\Gamma_2(\{2\}) &= \{1\} \\
\Gamma_2(\{1, 2\}) &= \{1, 2\}
\end{align*}
\]

Then \( X'_1 = \{1\} \) and \( X'_2 = \{1, 2\} \) are fixpoints of \( \Gamma_2 \).
Knaster-Tarski

Theorem (Knaster-Tarski)

Let $\Gamma$ be a monotonic operator on $X \neq \emptyset$. Then there is

- a unique least fixpoint
  $$\text{lfp}(\Gamma) = \bigcap \{M \subseteq X \mid \Gamma(M) = M\} = \bigcap \{M \subseteq X \mid \Gamma(M) \subseteq M\}$$
- a unique greatest fixpoint
  $$\text{gfp}(\Gamma) = \bigcup \{M \subseteq X \mid \Gamma(M) = M\} = \bigcup \{M \subseteq X \mid \Gamma(M) \supseteq M\}$$

Example (Fixpoint)

Let $X = \{1, 2\}$. Given the monotonic operator $\Gamma$ on $X$. Then $\text{lfp}(\Gamma_2) = \{1\}$ and $\text{gfp}(\Gamma_2) = \{1, 2\}$.

\[
\begin{align*}
\Gamma_2 (\emptyset) &= \{1\} \\
\Gamma_2 (\{1\}) &= \{1\} \\
\Gamma_2 (\{2\}) &= \{1\} \\
\Gamma_2 (\{1, 2\}) &= \{1, 2\}
\end{align*}
\]
Definition (Powers of a Monotonic Operator)

Let $\Gamma$ be a monotonic operator on $X \neq \emptyset$. Then

\[
\begin{align*}
\Gamma \uparrow 0 &= \emptyset \\
\Gamma \uparrow (\alpha + 1) &= \Gamma (\Gamma \uparrow \alpha) \\
\Gamma \uparrow \lambda &= \bigcup \{\Gamma \uparrow \beta \mid \beta < \lambda\} \\
\Gamma \downarrow 0 &= X \\
\Gamma \downarrow (\alpha + 1) &= \Gamma (\Gamma \downarrow \alpha) \\
\Gamma \downarrow \lambda &= \bigcap \{\Gamma \downarrow \beta \mid \beta < \lambda\}
\end{align*}
\]
Lemma

Let \( \Gamma \) be a monotonic operator on \( X \neq \emptyset \). Then

\[
\Gamma \uparrow \alpha \subseteq \Gamma \uparrow (\alpha + 1) \\
\Gamma \uparrow \alpha \subseteq \text{lfp} (\Gamma)
\]

If \( \Gamma \uparrow \alpha = \Gamma \uparrow (\alpha + 1) \) then \( \text{lfp} (\Gamma) = \Gamma \uparrow \alpha \)

Theorem

Let \( \Gamma \) be a monotonic operator on \( X \neq \emptyset \). Then \( \Gamma \uparrow \alpha = \text{lfp} (\Gamma) \) for some ordinal \( \alpha \).

Problem: In general uncountably many iterations required, i.e. impossible to compute!
Theorem (Kleene)

Let $\Gamma$ be a continuous operator on $X \neq \emptyset$. Then $\text{lfp} (\Gamma) = \Gamma \uparrow \omega$.

Example (Kleene)

To find a solution for a recursive equation with a continuous operator, e.g. $\Gamma (X) = X$, chaining can be applied:

```plaintext
X, X' := {}, {}
do
    X' := X
    X := gamma(X')
while X != X'
```
Difference between Knaster-Tarski and Kleene

- Kleene puts a stronger assumption on the operator.
- Kleene provides a computably enumerable method to calculate lfp.
- Knaster-Tarski provides information about the structure of fixpoints.
- Simple chaining using monotone operators is not computably enumerable in common.
Application in Logic Programming
Immediate Consequence Operator

Definition (Immediate Consequence Operator)

Let \( S \) be a set of universal generalized definite rules and \( B \subseteq HB \) a set of ground atoms. Then we define the **immediate consequence operator**

\[ T_S : \mathcal{P}(HB) \rightarrow \mathcal{P}(HB) \]

as

\[ T_S (B) = \{ A \in HB \mid \exists (A_1, \ldots, A, \ldots, A_n \leftarrow \phi) \in S : \text{HI}(B) \models \phi \} \]

Example

```
feeds_milk(betty) ← T
lays_eggs(betty) ← T
has_spines(betty) ← T

monotreme(X) ← lays_eggs(X), feeds_milk(X)
echidna(X) ← monotreme(X), has_spine(X)
blub(X) ← bla(X), echidna(X)
```

\[ T_S (\emptyset) = \{ \text{feeds}_\text{milk}(\text{betty}), \text{lays}_\text{eggs}(\text{betty}), \text{has}_\text{spines}(\text{betty}) \} \]

\[ T_S (\{ \text{monotreme}(\text{betty}), \text{has}_\text{spine}(\text{betty}) \}) = \{ \text{feeds}_\text{milk}(\text{betty}), \text{lays}_\text{eggs}(\text{betty}), \text{has}_\text{spines}(\text{betty}), \text{echidna}(\text{betty}) \} \]
Lemma ($T_S$ is continuous)

Let $S$ be a set of universal generalized definite rules. Then $T_S$ is continuous.

Example (Chaining $T_S$)

\[
F, \ F' := \{\}, \ {} \\
do \\
\quad F' := F \\
\quad F := T_S(F') \\
\text{while } F \neq F' 
\]

\[
F_0 = {} \\
F_1 = \{\text{feeds\_milk(betty), lays\_eggs(betty), has\_spines(betty)}\} \\
F_2 = \{\text{feeds\_milk(betty), lays\_eggs(betty), has\_spines(betty), monotreme(betty)}\} \\
F_3 = \{\text{feeds\_milk(betty), lays\_eggs(betty), has\_spines(betty), monotreme(betty), echidna(betty)}\} \\
F_4 = \{\text{feeds\_milk(betty), lays\_eggs(betty), has\_spines(betty), monotreme(betty), echidna(betty)}\}
\]
Theorem

Let $S$ be a set of universal generalized definite rules and $B \subseteq \text{HB}$ a set of ground atoms. Then

$$\text{HI}(B) \models S \iff T_S(B) \subseteq B$$

Corollary

Then \( \text{lfp}(T_S) = T_S \uparrow \omega = \text{Mod}_\cap(S) = \{ A \in \text{HB} \mid S \models A \} \). Then \( \text{HI}(\text{lfp}(T_S)) \) is the unique minimal Herbrand model of $S$.

We found a way to compute the "meaning" of universal definite rules!
Conclusion

1. Does the approach terminate?
   No, not in general!

2. Is there always a solution?
   Yes!

3. Is the solution unique?
   Yes!

4. What does the solution represent?
   The Unique Minimal Herbrand Model
Questions?