Definite Rules

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June 21, 2018
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Motivation
Theories can have many models, so if a model gives a meaning to a theory it can have many meanings.

Sometimes difference in models is relevant: example group theory
Example: group theory

A group \((G, e, \circ)\) consists of a set \(G\),
- a “neutral” element \(e \in G\)
- an operation \(\circ : G \times G \to G\)
- the identity relation
- the operation is associative
- \(e\) is the neutral element
- every element has an inverse

\[
\begin{align*}
\forall x, y, z ((x \circ y) \circ z &= x \circ (y \circ z)) \\
\forall x (e \circ x &= x = x \circ e) \\
\forall x \exists y (x \circ y &= e = y \circ x)
\end{align*}
\]

\(\text{Models:} \quad (\mathbb{Z}, 0, +)\)
\((\mathbb{R}, 0, +)\)
and also
\((\mathbb{R}, 1, \cdot)\)
permutations over \(\{a_1, \ldots, a_n\}\), with composition
invertible square matrices, with matrix product
polynomials, with addition
Example: group theory

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Models:

- \((\mathbb{Z}, 0, +)\)
- \((\mathbb{R}, 0, +)\) and also \((\mathbb{R}, 1, \cdot)\)
- permutations over \(\{a_1, \ldots, a_n\}\), with composition
- invertible square matrices, with matrix product
- polynomials, with addition
- ...

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\forall x, y, z ((x \circ y) \circ z = x \circ (y \circ z))
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\forall x (e \circ x = x = x \circ e)
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\forall x \exists y (x \circ y = e = y \circ x)
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Sometimes difference in models is relevant: example group theory.

But what are the relations between models, are there some special models?
Theories can have many models, so if a model gives a meaning to a theory it can have many meanings.

Sometimes difference in models is relevant: example group theory

But what are the relations between models, are there some special models?

Models can have undesired cardinality or be non normal (i.e. arbitrary interpretation of equality).

We want to see some of our intuitive constraints in the models:

- unique name assumption $\Rightarrow$ Herbrand models
- term constructor interpretation $\Rightarrow$ Herbrand models
- closed world assumption $\Rightarrow$ **minimal models**
Recap
Recap

polarity polarity of subformulas is a labeling of the syntax tree of a formula, start with positive, invert at every ¬ and on the left of ⇒.

rule written $\phi \leftarrow \psi$ stands for a universal closure $\forall^{*}(\psi \implies \phi)$ (also called clause)

def. rule special case of a rule/clause: $A \leftarrow B_1 \land \cdots \land B_n$

H. universe the set of all ground terms (variable free) over a language (needs at least one constant)

H. base the set of all ground atoms over a Herbrand universe, i.e. all atomic formulas without variables

H. model an interpretation/model with the Herbrand universe as domain

H. inducer subset of the Herbrand base that defines a Herbrand interpretation
Examples

Given a signature $L_1$ with $Fun^0 = \{a, b, c\}$, $Rel^1 = \{p, q\}$ we have

- $HU = Fun^0 = \{a, b, c\}$
- $HB = \{p(x) \mid x \in HU\} \cup \{q(x) \mid x \in HU\} = \{p(a), p(b), p(c), q(a), q(b), q(c)\}$

Some inducers:

- $I_1 = HI(B_1)$ for $B_1 = \{r(c, f(f(c))), r(f(c), f(f(f(c))))\} \subseteq HB$
- $I_2 = HI(B_2)$ for $B_2 = \{r(t_1, t_2) \mid t_2 \text{ has more } f \text{s than } t_1\} \subseteq HB$
Examples

Given a signature $\mathcal{L}_1$ with $\text{Fun}^0 = \{a, b, c\}$, $\text{Rel}^1 = \{p, q\}$ we have

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Given a signature $\mathcal{L}_\omega$ with $\text{Fun}^0 = \{c\}$, $\text{Fun}^1 = \{f\}$ and $\text{Rel}^2 = \{r\}$ we have

- $HU = \{c, f(c), f(f(c)), f(f(f(c))), f(f(f(f(c)))), \ldots\}$
- $HB = \{r(x, y) \mid x, y \in HU\} = \{r(c, c), r(c, f(c)), r(c, f(f(c))), \ldots, r(f(c), c), r(f(c), f(c)), \ldots\}$

Some inducers:

- $\mathcal{I}_1 = HI(B_1)$ for $B_1 = \{r(c, f(f(c))), r(f(c), f(f(f(c))))\} \subseteq HB$
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Definite rules and minimal models
Partial ordering of interpretations

Definition

For two interpretations $\mathcal{I}_1$, $\mathcal{I}_2$ we say $\mathcal{I}_1 \leq \mathcal{I}_2$ iff

- $\text{dom}(\mathcal{I}_1) = \text{dom}(\mathcal{I}_2)$,
- the interpretation of function symbols coincides,
- for all relation symbols $p$: $p^{\mathcal{I}_1} \subseteq p^{\mathcal{I}_2}$,
- and all variables are interpreted identically by $\mathcal{I}_1$ and $\mathcal{I}_2$. 

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Partial ordering of interpretations: Examples

Take the signature of $\mathcal{L}_\omega$ from above with

- $\mathcal{I}_1 = HI(B_1)$ for $B_1 = \{r(c, f(f(c))), r(f(c), f(f(f(c))))\} \subseteq HB$
- $\mathcal{I}_2 = HI(B_2)$ for $B_2 = \{r(t_1, t_2) \mid t_2 \text{ has more } fs \text{ than } t_1\} \subseteq HB$
- $\mathcal{I}_3 = HI(B_3)$ for $B_3 = \{r(t, t) \mid t \in HU\} \subseteq HB$
- $\mathcal{I}_4 = HI(B_4)$ for $B_4 = B_2 \cup B_3 \subseteq HB$

It follows:

- $\mathcal{I}_1 \leq \mathcal{I}_2$
- $\mathcal{I}_2 \leq \mathcal{I}_4$
- $\mathcal{I}_3 \leq \mathcal{I}_4$

But

- $\mathcal{I}_1 \not\leq \mathcal{I}_3$
- $\mathcal{I}_2 \not\leq \mathcal{I}_3$

Also for any interpretation $\mathcal{I} \leq \mathcal{I}$ and also $HI(\emptyset) \leq \mathcal{I} \leq HI(HB)$.
Intersection of interpretations

Definition

For some interpretations \((\mathcal{I}_i)_{i \in I}\) we say that they are **compatible** iff

- \(D = \bigcap_{i \in I} \text{dom}(\mathcal{I}_i) \neq \emptyset\)
- all \(\mathcal{I}_i\) interpret function symbols equally when restricted to \(D\)
- all interpretations interpret variables the same way

For compatible interpretations we define the **intersection** \(\mathcal{I} = \bigcap_{i \in I} \mathcal{I}_i\)

with

- \(\text{dom}(\mathcal{I}) = D\)
- \(f^\mathcal{I}(d_1, \ldots, d_n) = f^{\mathcal{I}_i}(d_1, \ldots, d_n)\) for some \(i\)
- \(p^\mathcal{I} = \bigcap_{i \in I} p^{\mathcal{I}_i}\)
- \(x^\mathcal{I} = x^{\mathcal{I}_i}\) for some \(i\)
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Intersection of interpretations: Examples

Take the language $\mathcal{L}_1$ from above with the formula

$$\phi = \forall x \exists y (p(x) \Rightarrow q(y))$$

Consider the inducers

$$B_1 = \{ p(a), q(b) \} \quad \text{HI}(B_1) \models \phi$$
$$B_2 = \{ p(a), q(c) \} \quad \text{HI}(B_2) \models \phi$$

Question: What is needed for the intersection of models of a theory to be a model of that theory again?
Intersection of interpretations: Examples

Take the language $\mathcal{L}_1$ from above with the formula

$$\phi = \forall x \exists y (p(x) \Rightarrow q(y))$$

Consider the inducers

$$B_1 = \{p(a), q(b)\} \quad HI(B_1) |= \phi$$
$$B_2 = \{p(a), q(c)\} \quad HI(B_2) |= \phi$$
$$B_1 \cap B_2 = \{p(a)\} \quad HI(B_1 \cap B_2) \not|= \phi$$
Intersection of interpretations: Examples

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$$B_1 = \{p(a), q(b)\} \quad HI(B_1) \models \phi$$
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$$B_1 \cap B_2 = \{p(a)\} \quad HI(B_1 \cap B_2) \not\models \phi$$
$$B_3 = \emptyset \quad HI(B_3) \models \phi$$

Question

What is needed for the intersection of models of a theory to be a model of that theory again?
Syntactical definitions

Definition (Positive and negative formula)
A formula is called **positive (negative)** iff every atomic subformula occurs in it with positive (negative) polarity.

Definition (Inductive formula)
If $\phi^+$ is a positive formula and $\phi^-$ is a negative formula we call

\[
\forall^* ( (A_1 \land \cdots \land A_n) \leftarrow \phi^+) \\
\forall^* \phi^-
\]

a generalised definite rule

and either of them an **inductive formula**.

Theorem

*Models of inductive theories are closed under intersection.*
Intersection of interpretations: Examples

Again with $\mathcal{L}_1$ consider the theory

$$T = \{ \exists x \, p(x), \, \forall x \, q(x) \}$$

then

$$C_1 = \{ p(a), p(b), q(a), q(b), q(c) \} \quad HI(C_1) \models T$$

$$C_2 = \{ p(b), p(c), q(a), q(b), q(c) \} \quad HI(C_2) \models T$$

$$C_3 = \{ p(a), p(c), q(a), q(b), q(c) \} \quad HI(C_3) \models T$$

$$C_4 = C_1 \cap C_2 = \{ p(b), q(a), q(b), q(c) \} \quad HI(C_4) \models T$$

$$C_5 = C_2 \cap C_3 = \{ p(c), q(a), q(b), q(c) \} \quad HI(C_5) \models T$$

$$C_6 = C_1 \cap C_3 = \{ p(a), q(a), q(b), q(c) \} \quad HI(C_6) \models T$$

$$C_7 = C_1 \cap C_2 \cap C_3 = \{ q(a), q(b), q(c) \} \quad HI(C_7) \not\models T$$

$$C_8 = \emptyset \quad HI(C_8) \not\models T$$
Intersection of interpretations: Examples

Again with $\mathcal{L}_1$ consider the theory

\[ T = \{ \exists x \ p(x), \ \forall x \ q(x) \} \]

then

\[ C_1 = \{ p(a), p(b), q(a), q(b), q(c) \} \quad HI(C_1) \models T \]
\[ C_2 = \{ p(b), p(c), q(a), q(b), q(c) \} \quad HI(C_2) \models T \]
\[ C_3 = \{ p(a), p(c), q(a), q(b), q(c) \} \quad HI(C_3) \models T \]
\[ C_4 = C_1 \cap C_2 = \{ p(b), q(a), q(b), q(c) \} \quad HI(C_4) \models T \]
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\[ C_6 = C_1 \cap C_3 = \{ p(a), q(a), q(b), q(c) \} \quad HI(C_6) \models T \]
\[ C_7 = C_1 \cap C_2 \cap C_3 = \{ q(a), q(b), q(c) \} \quad HI(C_7) \not\models T \]
\[ C_8 = \emptyset \quad HI(C_8) \not\models T \]
Intersection of interpretations: Examples

Again with $\mathcal{L}_1$ consider the theory

$$T = \{ \exists x \ p(x), \ \forall x \ q(x) \}$$

then

$$C_1 = \{ p(a), p(b), q(a), q(b), q(c) \} \quad HI(C_1) \models T$$
$$C_2 = \{ p(b), p(c), q(a), q(b), q(c) \} \quad HI(C_2) \models T$$
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$$C_5 = C_2 \cap C_3 = \{ p(c), q(a), q(b), q(c) \} \quad HI(C_5) \models T$$
$$C_6 = C_1 \cap C_3 = \{ p(a), q(a), q(b), q(c) \} \quad HI(C_6) \models T$$
$$C_7 = C_1 \cap C_2 \cap C_3 = \{ q(a), q(b), q(c) \} \quad HI(C_7) \not\models T$$
$$C_8 = \emptyset \quad HI(C_8) \not\models T$$
Question

What is needed for a minimal model to be unique?
Question
What is needed for a minimal model to be unique?

Theorem
If in an inductive theory either each formula is a generalised definite rule or if it is satisfiable and is universal then it has an unique minimal Herbrand model. For a theory of positive definite rules this is the intersection of all its Herbrand models.
Summary
We want our models to fulfill the **term constructor property**.

We want our models to fulfill the **closed world assumption**.
We want our models to fulfill the term constructor property.
⇒ Herbrand models

We want our models to fulfill the closed world assumption.
⇒ minimal models
Summary

- We want our models to fulfill the **term constructor property**.
  ⇒ **Herbrand models**

- We want our models to fulfill the **closed world assumption**.
  ⇒ **minimal models**

- We have a connection between some syntactical and some semantical properties of a theory.