# INSTITUT FÜR INFORMATIK der Ludwig-Maximilians-Universität München 

## Asymptotic Quantifier ELIMINATION

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## Bachelorarbeit

Hiermit versichere ich, dass ich die vorliegende Arbeit selbständig verfasst habe und keine anderen als die angegebenen Hilfsmittel verwendet habe.


#### Abstract

Quantifier elimination is a process of simplification, where for a formula $\phi$ with quantifiers a formula $\psi$ without quantifiers is constructed, which is equivalent to $\phi$. One adaptation of quantifier elimination is asymptotic quantifier elimination. Here one considers a probability measure $P_{n}$ over the set of finite models with universe size $n \in \mathbb{N}$ and a formula $\phi$ in first order predicate logic. The goal is to find a formula $\psi$ with no quantifiers such that $\lim _{n \rightarrow \infty} P_{n}(\phi \leftrightarrow \psi)=1$, where the limit is taken over the size of the models. A consequence of asymptotic quantifier elimination is the 0-1 Law for finite models, which states that for any closed formula $\phi$ in first order predicate logic without constants not function symbols $\lim _{n \rightarrow \infty} P_{n}(\phi) \in\{0,1\}$.

The work of Glebskii et. al. [7] proved, among other theorems, asymptotic quantifier elimination for finite models, when the formulas considered do not contain function symbols. The proof of the quantifier elimination theorem is constructive i.e. there is an algorithm which could be implemented in order to asymptotically eliminate the quantifiers of a formula. Independently and around the same time as Glebskii et. al. Fagin [5] proved the 0-1 Law for finite models in first order logic.

In this work we will expand on the work of Glebskii et. al. [7], by proving the asymptotic quantifier elimination theorem considering a more general probability measure. This result has been mentioned by [9], but to my knowledge has not been formally proven anywhere.


## Zusammenfassung

Quantorenelimination ist ein Prozess der Vereinfachung, wo für eine Formel $\phi$ mit Quantoren eine Formel $\psi$ ohne Quantoren erstellt wird, welche equivalent zu $\phi$ ist. Eine variante der Quantorenelimination ist Asymptotische Quantorenelimination. Hier werden ein Wahrscheinlichkeitsmass $P_{n}$ über die Menge der endlichen Modelle mit Universum der grösse $n \in \mathbb{N}$ und eine Formel $\phi$ in Predikatenlogik erster stufe betrachtet. Das ziel ist eine Formel $\psi$ ohne Quantoren zu finden s.d. $\lim _{n \rightarrow \infty} P_{n}(\phi \leftrightarrow \psi)=1$, wo der limis über die größe der modelle betrachtet wird. Eine Konsequenz der Asymptotischen Quantorenelimination ist das 0-1 Gesetz für endliche Modelle, welches besagt das für jede geschlossene Formel $\phi$ in Predikatenlogik erster stufe ohne Konstanten und Funktionssymbole $\lim _{n \rightarrow \infty} P_{n}(\phi) \in$ $\{0,1\}$.

Das Werk von Glebskii et. al. [7] bewies, unter anderen Sätzen, Asymptotischen Quantorenelimination für endliche Modelle, wenn die betrachteten Formeln keine Funktionssymbole beinhalten. Dieser Beweis ist Konstruktiv d.h. es existiert ein Algorithmus der implementiert werden kann um die quantoren einer Formel asymptotisch loszuwerden. Unabhängig und um denselben Zeitraum bewies Fagin [5] das 0-1 Gesetz für endliche Modelle in Predikatenlogik erster stufe.

In diesem Werk, werden wir die Arbeit von Glebskii et. al. [7] erweitern, indem die Asymptotischen Quantorenelimination für ein algemeineres Wahrscheinlichkeitsmass bewiesen wird. Dieses resultat wurde von [9] erwähnt, wurde aber zu meinem Wissen noch nicht formal bewiesen.

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Introduction

Quantifier elimination started with Tarski in 1931 [14], when he proved that the (semi)algebraic theory of $\mathbb{R}^{n}$ admitted quantifier elimination and developed an algorithm for it. Sadly the computational cost for this algorithm was impractical except for few instances. Seidenberg [13] and Cohen [3] provided alternative methods, which did not provide computational advantages compared to Tarski's. In 1975 Collins [4] provided a new method, which has a doubly exponential complexity in the number of variables. This was a huge improvement compared to other methods [1].

Further theories for which quantifier elimination has been proven, include Presburger Arithmetic [11], algebraically closed fields [2], atomless boolean algebras [10], among multiple others. Some of the interest in proving that a theory allows for quantifier elimination is to consider only quantifier free formulas when proving some statement. Whether a theory is complete and/or decidable are some properties for which this can be done.

In this thesis, we will consider quantifier elimination not in the context presented originally by Tarski [14], in which formulas must be equivalent no matter the interpretation, but in the context presented by Glebskii et. al. [7]. They consider quantifier elimination for a formula $\phi$ in most of the possible interpretations. This means there exists a formula $\psi$ without quantifiers such that in most of the models $\phi \leftrightarrow \psi$ is a tautology. In contrast to the (semi-)algebraic theory of $\mathbb{R}^{n}$ an algorithm for this question can be found in PSPACE and the problem itself is PSPACE complete [8]. More formally, given a probability measure $P_{n}$ over the set of finite models with universe size $n \in \mathbb{N}$ and a formula $\phi$ in first order predicate logic the task of quantifier elimination, under consideration, is to find a formula $\psi$ with no quantifiers such that $\lim _{n \rightarrow \infty} P_{n}(\phi \leftrightarrow \psi)=1$, where the limit is taken over the size of the models ${ }^{1}$. This means, a model fulfills $\psi \leftrightarrow \phi$ with a probability which increases with model size. This implies that the probability that a model of size $n$ fulfills $\psi$ gets closer, with increasing model size, to the probability that a model fulfills $\phi$. In other words, we can approximate the probability with which $\phi$ is "true" by determining the probability that $\psi$ is "true".

Determining with which probability a formula is "true" is an interesting question in statistical relational AI, where given some amount of data points and/or underlying assumptions, one wants to know what the probability is that some formula $\phi$ in first order logic is "true". In order to do this, multiple algorithms have been developed [6, 8] and

[^0]programming languages created [12].
One possible way to increase the performance of these algorithms is to make them independent of the model under consideration, which can be done using quantifier elimination. By removing all quantifiers, all remaining variables are unbound, which means that evaluating if a model fulfills the formula is independent of the size of the model and only depends on the assignment of the free variables. This allows one to approximate the probability that a formula $\phi$ is "true" by determining the probability that a formula $\psi$ without quantifiers is "true". For example in the formula $r(y) \vee \forall x(r(x))$ the quantified term $\forall x(r(x))$ is only true in a single model for every model size $n$. This means, assuming $P_{n}$ to be the uniform distribution, the probability that $\forall x(r(x))$ is "true" is $\frac{1}{2^{n}}$, for a model size of $n$, which converges to zero exponentially fast. This means the probability that $r(y) \vee \forall x(r(x))$ is "true" can be approximated by $r(y)$ with an error which vanishes exponentially fast with model size. One way to find such a formula $\psi$ is by the quantifier elimination theorem as presented by Glebskii et. al. [7]. Using quantifier elimination would allow an algorithm to use a prepossessing step, where the formula $\phi$ is compiled into a formula $\psi$ which can then be efficiently evaluated.

The work of Glebskii et. al. proved, among other theorems, quantifier elimination and as consequence the 0-1 Law for finite models, when the formulas considered do not contain function symbols. The proof of the quantifier elimination theorem is constructive i.e. we have an algorithm which could be implemented in order to eliminate the quantifiers of a formula. Independently and around the same time as Glebskii et. al. Fagin [5] proved the 0-1 Law for finite models in first order logic. Fagin proved the 0-1 Law by introducing a set $T$ of extensionality axioms and proving that $T$ is complete and consistent in first order logic. Afterwards he proved that the 0-1 Law is valid for all extensionality axioms and generalized this to all formulas, which can be done since $T$ is complete.

In this work we will expand on the work of Glebskii et. al. [7], by proving the quantifier elimination theorem considering a more general probability measure. This result has been mentioned by [9], but to my knowledge has not been formally proven anywhere.

First we will give some background and notation required to understand the rest of the text. This will include first order logic and some probability theory. Afterwards we will proceed to prove the quantifier elimination theorem. This will be achieved by introducing exclusive quantifiers, and proving that closed formulas containing only exclusive quantifiers obey the 0-1 Law. We will also show that every formula can be translated into a formula with only exclusive quantifiers. Afterwards it will be proven that every formula in first order logic can be divided into parts containing only free variables and parts where at least one variable is bound. The parts with at least one bound variable obey the 0-1 Law and are therefore equivalent to either "true" or "false". This means they can be ignored when the model size goes to infinity. This brief description of the proof will be formalized in the following sections.

## CHAPTER 2

Background

### 2.1 First Order Predicate Logic

Notation 2.1.1. In the following, let $L=(V, C, R)$ be a language with a countably infinite set of variable symbols $V$, a finite set of constant symbols $C$ and a finite set of predicate symbols $R$, which contains $=$ i.e. a symbol for equality. Each predicate symbol $r \in R$ is associated with an arity $\operatorname{ar}(r) \in \mathbb{N}$ and $=$ has arity 2 . Except otherwise stated, $C=\emptyset$ and we will write $L=(V, R)^{1}$

Definition 2.1.2. Terms are defined by:

- Every variable is a term.
- Every constant is a term.

Definition 2.1.2 does not consider composite terms i.e. terms with function symbols, since, in general, the results presented do not apply for formulas containing them.

Definition 2.1.3. Formulas are defined by:

- If $t_{1}, \cdots, t_{n}$ are terms and $p$ a predicate symbol with arity $n$, then $p\left(t_{1}, \cdots, t_{n}\right)$ is a formula.
- If $\phi$ and $\psi$ are formulas, then $(\phi \wedge \psi),(\phi \vee \psi),(\phi \rightarrow \psi),(\phi \leftrightarrow \psi)$ and $(\neg \phi)$ are formulas.
- If $\phi$ is a formula and $x$ a variable, then $\forall x(\phi)$ and $\exists x(\phi)$ are formulas.

Parenthesis can be left out if they are clear from context.
Notation 2.1.4. In the following work, $\Pi$ will denote either $\forall$ or $\exists$.
Notation 2.1.5. We write $\Pi_{1} x_{1}, \cdots, \Pi_{m} x_{m} \phi$ for $\Pi_{1} x_{1}\left(\Pi_{2} x_{2}\left(\cdots\left(\Pi_{m} x_{m}(\phi)\right)\right)\right)$.
Notation 2.1.6. We write $\phi=\psi$ if the formulas $\phi$ and $\psi$ are syntactically equal and $\phi \neq \psi$ if they are not.

Notation 2.1.7. We write $\bigwedge_{i \in I} \phi_{i}$ for $\left(\left(\left(\phi_{i_{1}} \wedge \phi_{i_{2}}\right) \wedge \cdots\right) \wedge \phi_{i_{m}}\right)$ and $\bigvee_{i \in I} \phi_{i}$ for $\left(\left(\left(\phi_{i_{1}} \vee \phi_{i_{2}}\right) \vee \cdots\right) \vee\right.$ $\left.\phi_{i_{m}}\right)$, where $I=\left\{i_{1}, i_{2}, \cdots, i_{m}\right\}$ is a finite index set.

[^1]Definition 2.1.8. The free variables of a formula $\phi \mathrm{fv}(\phi)$ are defined by:

- $\mathrm{fv}(\phi \wedge \psi)=\mathrm{fv}(\phi \vee \psi)=\mathrm{fv}(\phi \rightarrow \psi)=\mathrm{fv}(\phi \leftrightarrow \psi)=\mathrm{fv}(\phi) \cup \mathrm{fv}(\psi)$.
- $\operatorname{fv}(\neg \phi)=\operatorname{fv}(\phi)$.
- $\operatorname{fv}(\Pi x \phi)=\operatorname{fv}(\phi) \backslash\{x\}$.
- $\left.\operatorname{fv}\left(p\left(t_{1}, \cdots, t_{n}\right)\right)\right)=\bigcup_{i=1}^{n} \mathrm{fv}\left(t_{i}\right)$ for a predicate symbol $p$ with arity $n$ and terms $t_{1}, \cdots, t_{n}$.
- $\mathrm{fv}(x)=x$ for a variable $x$.
- $\mathrm{fv}(c)=\emptyset$ for a constant $c$.

Definition 2.1.9. The substitution of the term $t$ for the variable $x$ in the formula $\phi$ i.e. $\phi[x \mapsto t]$ is defined by:

- If $y$ is a variable, then $y[x \mapsto t]$ equals $t$ if $y=x$ and $y$ if $y \neq x$.
- If $c$ is a constant, then $c[x \mapsto t]$ equals $c$.
- $p\left(t_{1}, \cdots, t_{n}\right)[x \mapsto t]$ equals $p\left(t_{1}[x \mapsto t], \cdots, t_{n}[x \mapsto t]\right)$.
- $\left(\phi_{1} \circ \phi_{2}\right)[x \mapsto t]$ equals $\left(\phi_{1}[x \mapsto t]\right) \circ\left(\phi_{2}[x \mapsto t]\right)$ if $\circ$ is one of $\wedge, \vee, \rightarrow, \leftrightarrow$.
- $(\neg \psi)[x \mapsto t]$ equals $\neg(\psi[x \mapsto t])$.
- $\left(\Pi y \phi_{1}\right)[x \mapsto t]$ equals $\Pi y \phi_{1}$ if $x=y$ and $\Pi y\left(\phi_{1}[x \mapsto t]\right)$ if $x \neq y$.

Definition 2.1.10. A model $M=(U, K, S)$ for a language $L=(V, C, R)$, is defined as an interpretation of formulas with a universe $U=\{1, \cdots, n\}$ for some $n \in \mathbb{N}$. This means that for every constant $c \in C$ there exists an interpretation $K(c) \in U$ and for every predicate symbol $p \in R$ with arity $k$ there exists an interpretation $S(p) \subseteq U^{k}$. We write $M=(U, S)$ if $L=(V, R)$ i.e. $C=\emptyset$. We define $\mathscr{M}_{n}$ as all models with the universe $\{1, \cdots, n\}$.

Example 2.1.11. There exist $2^{n^{k}}$ models with $M=(\{1, \cdots, n\}, S)$ given a Language $L=(V,\{p\})$, with $p$ a predicate symbol with arity $k \in \mathbb{N}$.

Definition 2.1.12. Let $M=(U, K, S)$ be a model and $\phi$ a formula, then a function $\mu: \mathrm{fv}(\phi) \mapsto U$ is called a variable asssignment for (the variables of) the formula $\phi$ with respect to the model $M$.

Definition 2.1.13. Let $M=(U, K, S)$ be a model, $\phi$ a formula and $\mu$ a variable assignment for the formula $\phi$ and model $M$. The relation $\vDash$ as defined below is called a models relation and written as $M \models{ }_{\mu} \phi$ if $(M, \mu, \phi) \in \models$.

- $M \models{ }_{\mu} p\left(t_{1}, \cdots, t_{n}\right)$ if and only if $\left(\sigma_{1}\left(t_{1}\right), \cdots, \sigma_{n}\left(t_{n}\right)\right) \in S(p)$ whenever $\sigma_{i}=\mu$ if $t_{i}$ is a variable and $\sigma_{i}=K$ if $t_{i}$ is a constant.
- $M \models{ }_{\mu} t_{1}=t_{2}$ if and only if $\sigma_{1}\left(t_{1}\right)=\sigma_{2}\left(t_{2}\right)$ whenever $\sigma_{i}=\mu$ if $t_{i}$ is a variable and $\sigma_{i}=K$ if $t_{i}$ is a constant.
- $M \models_{\mu} \neg \phi$ if and only if is is not the case that $M \models_{\mu} \phi$.
- $M \models_{\mu} \phi \wedge \psi$ if and only if $M \models_{\mu} \phi$ and $M \models_{\mu} \psi$.
- $M \models_{\mu} \phi \vee \psi$ if and only if $M \models_{\mu} \phi$ or $M \models_{\mu} \psi$.
- $M \models_{\mu} \phi \rightarrow \psi$ if and only if $M \models_{\mu} \phi$ implies $M \models_{\mu} \psi$.
- $M \models_{\mu} \phi \leftrightarrow \psi$ if and only if $M \models_{\mu} \phi \rightarrow M \models_{\mu} \psi$ and $M \models_{\mu} \psi \rightarrow M \models_{\mu} \phi$.
- $M \models_{\mu} \forall x \phi$ if and only iffor all $x_{0} \in U$ it is the case that $M \models_{\mu} \phi\left[x \mapsto x_{0}\right]$.
- $M \models_{\mu} \exists x \phi$ if and only if there exists an $x_{0} \in U$ such that $M \models_{\mu} \phi\left[x \mapsto x_{0}\right]$.

If $M=_{\mu} \phi$ for all variable assignments $\mu$ then we write $M \models \phi$. In particular, if $\operatorname{fv}(\phi)=\emptyset$ then for every variable assignment $\mu, M \models_{\mu} \phi$ if and only if $M \models \phi$. We write $M \not \models_{\mu} \phi$ if it is not the case that $M \mid={ }_{\mu} \phi$.

Definition 2.1.14. We write $\models \phi$ and call $\phi$ a tautology if $M \models \phi$ for all models $M$.
Notation 2.1.15. We write $\phi \equiv \psi$ if $\phi \leftrightarrow \psi$ is a tautology and $\phi \not \equiv \psi$ if it is not.
Definition 2.1.16. A formula $\phi=\Pi_{1} x_{1} \cdots \Pi_{n} x_{n} \psi$ is said to be in prenex normal form, if $\psi$ contains no quantifiers.

Definition 2.1.17. A formula $\phi$ is in negation normal form (nnf) if every negation symbol is directly before a predicate symbol.

Example 2.1.18. $\neg p(x)$ is in nnf, while $\neg(p(x) \wedge q(x))$ is not
Definition 2.1.19. Let $t_{1}, \cdots, t_{n}$ be terms and $p$ a predicate symbol with arity $n$. Then $p\left(t_{1}, \cdots, t_{n}\right)$ is called an atomic formula.

Definition 2.1.20. Let $\phi$ be an atomic formula, then $\phi$ and $\neg \phi$ are called literals.
Definition 2.1.21. A formula $\phi$ is in disjunctive normal form (dnf) if $\phi=\bigvee_{i \in I} \bigwedge_{j \in J_{i}} \phi_{i j}$ where $\phi_{i j}$ is a literal, for some finite index sets $I, J_{i}$.

Example 2.1.22. $(p(x) \wedge q(x)) \vee \neg q(x)$ is in dnf, while $\neg(p(x) \wedge q(x))$ is not
Lemma 2.1.23. For every formula $\phi$ without quantifiers there exists a formula $\psi$ without quantifiers in disjunctive normal form such that $\phi \equiv \psi$.

Proof. We consider a formula $\phi$, where the symbols $\psi \rightarrow \theta$ and $\psi \leftrightarrow \theta$ have been replaced by $\neg \psi \vee q$ and $(\neg p \vee q) \wedge(\neg q \vee p)$ respectively. The following algorithm can be used to find the disjunctive normal form of $\phi$.

Listing 2.1: Algorithm for building the disjunctive normal form of a formula

```
def build_negation_normal_form(phi):
    if phi == not (p or q):
        return (not p) and (not q)
    elif phi == not (p and q):
        return (not p) or (not q)
    elif phi == not not p:
        return p
    else:
        return phi
def build_dnf_rec(phi):
    if phi == p or q:
        return build_dnf_rec(p) or build_dnf_rec(q)
    elif phi == p and (q or r):
        s = build_dnf_rec(p)
        return (s and build_dnf_rec(q)) or (s and build_dnf_rec(r))
    elif phi == (q or r) and p:
        s = build_dnf_rec(p)
```

```
        return (s and build_dnf_rec(q)) or (s and build_dnf_rec(r))
    else:
    return phi
def build_dnf(phi):
    psi = build_negation_normal_form(phi)
    new_psi = build_dnf_rec(psi)
    while new_psi != psi:
        psi = new_psi
        new_psi = build_dnf_rec(psi)
    return psi
```

This algorithm will terminate, since $\vee$ is propagated above $\wedge$, which means that at some point no further iteration can be done, since no more elements of the form $p \wedge(q \vee r)$ remain.

Definition 2.1.24. A formula $\phi$ is in disjoint disjunctive normal form (ddnf) if $\phi=\bigvee_{i \in I} \wedge_{j \in J_{i}} \phi_{i j}$ where each $\phi_{i j}$ is a literal, and for all $i, k \in I$ with $i \neq k \neg\left(\bigwedge_{j \in J_{i}} \phi_{i j} \wedge \bigwedge_{j \in J_{k}} \phi_{k j}\right)$ is a tautology.

Example 2.1.25. $(p(x) \wedge q(x)) \vee(p(x) \wedge \neg q(x))$ is in ddnf, while $(p(x) \wedge q(x)) \vee \neg(p(x) \wedge r(x))$ is in dnf but not in ddnf.
Lemma 2.1.26. For every formula $\phi$ without quantifiers there exists a formula $\psi$ in ddnf such that $\phi \equiv \psi$.
Proof. Let $\phi_{1}=\bigvee_{i \in I} \bigwedge_{j \in J_{i}} \phi_{i j}$ be the dnf of $\phi$. Then $\psi$, as defined in Equation 2.1, is the ddnf form of $\phi$, where $S=\left\{\phi_{i j} \mid i \in I, j \in J_{i}\right\}$.

$$
\begin{equation*}
\psi=\bigvee_{i \in I}\left(\bigwedge_{j \in J_{i}} \phi_{i j} \wedge \bigwedge_{\sigma \in S \backslash \bigcup_{j \in J_{i}}\left\{\phi_{i j}\right\}} \neg \sigma\right) \tag{2.1}
\end{equation*}
$$

In order to provide a link between Glebskii et. al. and Fagin's work on the 0-1 Law the following definition of an extensionality axiom is needed.
Definition 2.1.27. Let $R$ be the set of all predicate symbols in the language considered, then a $t+1$ extensionality axiom is given by

$$
\begin{equation*}
\forall x_{1}, \cdots, x_{l}\left(\bigwedge_{1 \leq i<j \leq l} x_{i} \neq x_{j} \rightarrow \exists x_{l+1}\left(\bigwedge_{1 \leq i \leq l} x_{i} \neq x_{l+1} \wedge \bigwedge_{\phi \in \Phi} \phi \wedge \bigwedge_{\phi \in \Phi^{C}} \neg \phi\right)\right) \tag{2.2}
\end{equation*}
$$

Where $\Phi \subseteq \Delta_{r+1}$ and $\Phi^{C}=\Delta_{r+1} \backslash \Phi$, with:

$$
\begin{equation*}
\Delta_{r+1}=\left\{r\left(z_{1}, \cdots, z_{m}\right) \mid x_{r+1} \in\left\{z_{1}, \cdots, z_{m}\right\} \wedge\left\{z_{1}, \cdots, z_{m}\right\} \subseteq\left\{x_{1}, \cdots, x_{r+1}\right\} \wedge r \in R\right\} \tag{2.3}
\end{equation*}
$$

### 2.2 Probability Measure

Definition 2.2.1. A probability measure is a function $P: \mathscr{P}(\Omega) \rightarrow[0,1]$ over a finite set $\Omega$ that satisfies the following properties:

$$
\begin{equation*}
P(\Omega)=1 \tag{2.4}
\end{equation*}
$$

$$
\begin{equation*}
\text { For }\left\{E_{i}\right\}_{i \in \mathbb{N}} \text { pairwise disjoint sets } P\left(\cup_{i \in \mathbb{N}} E_{i}\right)=\sum_{i \in \mathbb{N}} P\left(E_{i}\right) \tag{2.5}
\end{equation*}
$$

Lemma 2.2.2. A probability measure $P: \mathscr{P}(\Omega) \rightarrow[0,1]$, with $A, B \subseteq \Omega$ has the following properties:
a) $P(A)=1-P(\Omega \backslash A)$.
b) $P(A \cup B)=P(A)+P(B)-P(A \cap B)$ and specially $P(A \cup B) \leq P(A)+P(B)$.
c) If $A \subseteq B$ then $P(A) \leq P(B)$.

Proof. Let $A, B \subseteq \Omega$.
a) Since $A \cap(\Omega \backslash A)=\varnothing$ it follows that $1=P(\Omega)=P(A \cup(\Omega \backslash A))=P(A)+P(\Omega \backslash A)$. From this it follows that $P(A)=1-P(\Omega \backslash A)$.
b) Using a) and $(A \backslash(A \cap B)) \cap B=\varnothing$ we see that:

$$
\begin{align*}
& P(A \cup B)=P((A \backslash(A \cap B)) \cup B)=P(A \backslash(A \cap B))+P(B)= \\
& 1-P((\Omega \backslash A) \cup(A \cap B))+P(B)=1-P(\Omega \backslash A)-P(A \cap B)+P(B)= \\
& \quad P(A)-P(A \cap B)+P(B) \leq P(A)+P(B) . \tag{2.6}
\end{align*}
$$

c) Since $A \cup(B \backslash A)=B, A \cap(B \backslash A)=\emptyset$ and $P(B \backslash A) \geq 0$ it follows that $P(A) \leq P(A)+P(B \backslash A)=$ $P(A \cup(B \backslash A))=P(B)$.

## chapter 3

## Quantifier Elimination

Quantifier elimination can be used in mathematical logic, model theory, theoretical computer science and statistical relational AI in order to simplify sentences. An example application of this is to answer the question " $\exists x \phi$ " i.e. "When is there an x such that $\phi$ ", since the statement without quantifiers can be viewed as the answer to that question [15]. For example consider the sentence $\exists x(r(x))$ then the sentence without quantifiers for a model size of $n$ is $\bigvee_{i=1}^{n} r(i)$. In other words if we ask "Is there an $x$ such that $r(x)$ " the answer to this question for a model size of $n$ is "Either $r(1)$ or $r(2)$ or $\cdots$ or $r(n)$ ". This characterization of quantifier elimination can then be applied to approximations, where the question "When is there an $x$ such that $\phi$ " needs an answer "In most cases when $\psi$ ". This second characterization will be formalized in this section, where we are concerned with quantifier elimination concerning the relation $\models$, when the model size goes to infinity.

Definition 3.1.1. Let $P_{n}: \mathscr{M}_{n} \rightarrow(0,1)$ be a probability measure over the set of all models with $n$ elements. It assigns to each predicate symbol $r$ a probability $\sigma_{r} \in(0,1)$, independent of $n$ and of the probabilities of other predicate symbols. This means that for all $y_{1}, \cdots, y_{\operatorname{ar}(r)} \in\{1, \cdots, n\}$ with $\mu\left(x_{i}\right)=y_{i}$ :

$$
\begin{equation*}
P_{n}\left(\left\{M \in \mathscr{M}_{n}: M=_{\mu} r\left(x_{1}, \cdots, x_{\operatorname{ar}(r)}\right)\right\}\right)=\sigma_{r} \tag{3.1}
\end{equation*}
$$

and for any $\left\{r_{i}\right\}_{i \in \mathbb{N}}$, with $r_{i}\left(x_{i 1}, \cdots, x_{\operatorname{iar}(r)}\right) \neq r_{j}\left(x_{j 1}, \cdots, x_{j \operatorname{ar}(r)}\right)$ for $i \neq j$, it is the case that:

$$
\begin{equation*}
P_{n}\left(\bigcap_{i \in \mathbb{N}}\left\{M \in \mathscr{M}_{n}: M \models{ }_{\mu} r_{i}\left(x_{i 1}, \cdots, x_{i \operatorname{ar}(r)}\right)\right\}\right)=\prod_{i \in \mathbb{N}} P_{n}\left(\left\{M \in \mathscr{M}_{n}: M \models_{\mu} r_{i}\left(x_{i 1}, \cdots, x_{i \operatorname{ar}(r)}\right)\right\} .\right. \tag{3.2}
\end{equation*}
$$

We write $P_{n}(\phi, \mu)$ for $P_{n}\left(\left\{M \in \mathscr{M}_{n}: M \models{ }_{\mu} \phi\right\}\right)$.

Definition 3.1.1 is dependent on the variable assignment used. To remove this dependency we prove the following Lemma.

Lemma 3.1.2. Let $P_{n}$ be a probability measure as given by Definition 3.1.1 and $\mu_{1}$ and $\mu_{2}$ be two variable assignments such that $\mu_{1}(x)=\mu_{1}(y) \leftrightarrow \mu_{2}(x)=\mu_{2}(y)$ then $P_{n}\left(\phi, \mu_{1}\right)=P_{n}\left(\phi, \mu_{2}\right)$.

Proof. Let $\psi$ be the ddnf of $\phi$, according to Lemma 3.1.6 and Definition 3.1.1 it follows that:

$$
\begin{align*}
P_{n}\left(\phi, \mu_{1}\right)=P_{n}\left(\psi, \mu_{1}\right) & =P_{n}\left(\bigvee_{i \in I} \bigwedge_{j \in J_{i}} r_{i j}\left(x_{i j 1}, \cdots, x_{i j \operatorname{ar}\left(r_{i j}\right)}\right), \mu_{1}\right)  \tag{3.3}\\
& =\sum_{i \in I} P_{n}\left(\bigwedge_{j \in J_{i}} r_{i j}\left(x_{i j 1}, \cdots, x_{i j \operatorname{ar}\left(r_{i j}\right)}\right), \mu_{1}\right)  \tag{3.4}\\
& =\sum_{i \in I} \prod_{j \in J_{i}} P_{n}\left(r_{i j}\left(x_{i j 1}, \cdots, x_{i j \operatorname{ar}\left(r_{i j}\right)}\right), \mu_{1}\right)  \tag{3.5}\\
& =\sum_{i \in I} \prod_{j \in J_{i}} P_{n}\left(r_{i j}\left(x_{i j 1}, \cdots, x_{i j \operatorname{ar}\left(r_{i j}\right)}\right), \mu_{2}\right)  \tag{3.6}\\
& =\sum_{i \in I} P_{n}\left(\bigwedge_{j \in J_{i}} r_{i j}\left(x_{i j 1}, \cdots, x_{i j \operatorname{ar}\left(r_{i j}\right)}\right), \mu_{2}\right)  \tag{3.7}\\
& =P_{n}\left(\bigvee_{i \in I} \bigwedge_{j \in J_{i}} r_{i j}\left(x_{i j 1}, \cdots, x_{i j \operatorname{ar}\left(r_{i j}\right)}\right), \mu_{2}\right)  \tag{3.8}\\
& =P_{n}\left(\psi, \mu_{2}\right)=P_{n}\left(\phi, \mu_{2}\right) \tag{3.9}
\end{align*}
$$

Remark 3.1.3. In Lemma 3.1.2 the requirement that $\mu_{1}(x)=\mu_{1}(y) \leftrightarrow \mu_{2}(x)=\mu_{2}(y)$ is needed, since for example for all $M M={ }_{\mu} r(x) \wedge r(y) \leftrightarrow r(x)$ if $\mu(x)=\mu(y)$, but in general $P_{n}(r(x) \wedge r(y)) \neq$ $P_{n}(r(x))$.

Notation 3.1.4. Since in every result the variable assignments used do not assign two variables to the same value $P_{n}$ is independent of the variable assignment used. Therefore we write $P_{n}(\phi)$ instead of $P_{n}(\phi, \mu)$ except when a specific variable assignment is considered.

The last step needed for Definition 3.1.1 is to prove the uniqueness of the probability measure, since the existence is clear as $\mathscr{M}_{n}$ is finite.

Lemma 3.1.5. Let $P_{n}$ and $P_{n}^{\prime}$ be two probability measures as given by Definition 3.1.1 such that $P_{n}\left(r\left(x_{1}, \cdots, x_{m}\right)\right)=P_{n}^{\prime}\left(r\left(x_{1}, \cdots, x_{m}\right)\right)$ for every predicate symbol $r$. Then $P_{n}=P_{n}^{\prime}$.

Proof. Let $M \subseteq \mathscr{M}_{n}$ and let $L=(V, C, R)$ be the language under consideration. For the formula $\phi$, described in equation 3.10, it is the case that $P_{n}(\phi)=P_{n}(M)$, since every model is uniquely determined by the assignment of predicate and constant symbols.

$$
\begin{gather*}
\phi=\bigvee_{(U, K, S) \in M} \psi_{(U, K, S)}  \tag{3.10}\\
\psi_{(U, K, S)}=\bigwedge_{r \in R}\left(\bigwedge_{s \in S(r)} r(s) \wedge \bigwedge_{s \in U^{\text {arr }(r)} \text { \S(r)}} \neg r(s)\right) \wedge \bigwedge_{c \in C \backslash \mathbb{N}} c=K(c) \tag{3.11}
\end{gather*}
$$

If $(U, K, S) \models_{\mu} \psi_{(U, K, S)}$ and $\left(U^{\prime}, K^{\prime}, S^{\prime}\right)=_{\mu} \psi_{\left(U^{\prime}, K^{\prime}, S^{\prime}\right)}$ with $(U, K, S),\left(U^{\prime}, K^{\prime}, S^{\prime}\right) \in M$ then $(U, K, S)=$ $\left(U^{\prime}, K^{\prime}, S^{\prime}\right)$. This means that $\left\{M: M \models_{\mu} \psi_{(U, K, S)}\right\} \cap\left\{M: M=_{\mu} \psi_{\left(U^{\prime}, K^{\prime}, S^{\prime}\right)}=\emptyset\right.$ for $(U, K, S) \neq$
$\left(U^{\prime}, K^{\prime}, S^{\prime}\right)$. From this it follows by Definition 2.2.1 and Definition 3.1.1 that:

$$
\begin{align*}
P_{n}(M) & =P_{n}(\phi)  \tag{3.12}\\
& =\sum_{(U, K, S) \in M} P_{n}\left(\bigwedge_{r \in R}\left(\bigwedge_{s \in S(r)} r(s) \wedge \bigwedge_{s \in U^{\operatorname{ara}(r)} \backslash S(r)} \neg r(s)\right) \wedge \bigwedge_{c \in C \backslash \mathbb{N}} c=K(c)\right)  \tag{3.13}\\
& =\sum_{(U, K, S) \in M} \prod_{r \in R}\left(\prod_{s \in S(r)} P_{n}(r(s)) \times \prod_{s \in U^{\operatorname{ar}(r) \backslash S(r)}}\left(1-P_{n}(r(s))\right)\right) \times \prod_{c \in C \backslash \mathbb{N}} P_{n}(c=K(c))  \tag{3.14}\\
& =\sum_{(U, K, S) \in M} \prod_{r \in R}\left(\prod_{s \in S(r)} P_{n}^{\prime}(r(s)) \times \prod_{s \in U^{\operatorname{ara}(r) \backslash S(r)}}\left(1-P_{n}^{\prime}(r(s))\right)\right) \times \prod_{c \in C \backslash \mathbb{N}} P_{n}^{\prime}(c=K(c))  \tag{3.15}\\
& =\sum_{(U, K, S) \in M} P_{n}^{\prime}\left(\bigwedge_{r \in R}\left(\bigwedge_{s \in S(r)} r(s) \wedge \bigwedge_{s \in U^{\operatorname{ar}(r) \backslash S(r)}} \neg r(s)\right) \wedge \bigwedge_{c \in C \backslash \mathbb{N}} c=K(c)\right)  \tag{3.16}\\
= & P_{n}^{\prime}(\phi)  \tag{3.17}\\
= & P_{n}^{\prime}(M) \tag{3.18}
\end{align*}
$$

From this it follows that $P_{n}=P_{n}^{\prime}$.
Lemma 3.1.6. Let $P_{n}$ be defined as in 3.1.1 and $\phi, \psi$ be formulas, then:
a) If $\phi \equiv \psi$ then $P_{n}(\phi)=P_{n}(\psi)$.
b) $P_{n}(\phi)=1-P_{n}(\neg \phi)$.
c) $P_{n}(\phi \vee \psi)=P_{n}(\phi)+P_{n}(\psi)-P(\phi \wedge \psi)$, in particular $P_{n}(\phi \vee \psi) \leq P_{n}(\phi)+P_{n}(\psi)$.
d) If $\phi \rightarrow \psi$ is a tautology then $P_{n}(\phi) \leq P_{n}(\psi)$.

Proof. Let $\phi, \psi$ be formulas, then:
a) Follows from Definitions 3.1.1 and 2.1.14
b) It is the case that $1=P\left(\mathscr{M}_{n}\right)=P_{n}\left(\left\{M \in \mathscr{M}_{n} \mid M \models_{\mu} \phi\right\} \cup\left\{M \in \mathscr{M}_{n} \mid M \models_{\mu} \neg \phi\right\}\right)=P_{n}(\phi)+$ $P_{n}(\neg \phi)$, since

$$
\begin{align*}
& \left\{M \in \mathscr{M}_{n}|M|=_{\mu} \phi\right\} \cup\left\{M \in \mathscr{M}_{n} \mid M \models_{\mu} \neg \phi\right\}=\mathscr{M}_{n}  \tag{3.19}\\
& \left\{M \in \mathscr{M}_{n} \mid M=_{\mu} \phi\right\} \cap\left\{M \in \mathscr{M}_{n} \mid M \models_{\mu} \neg \phi\right\}=\varnothing . \tag{3.20}
\end{align*}
$$

From Lemma 2.2.2 it follows that $P_{n}(\phi)=1-P_{n}(\neg \phi)$.
c) It is the case that:

$$
\begin{equation*}
\left.\left\{M \in \mathscr{M}_{n} \mid M \models_{\mu} \phi\right)\right\} \cup\left\{M \in \mathscr{M}_{n} \mid M \models_{\mu} \psi\right\}=\left\{M \in \mathscr{M}_{n} \mid M \models_{\mu} \phi \vee \psi\right\} . \tag{3.21}
\end{equation*}
$$

From Lemma 2.2.2 it follows that $P_{n}(\phi \vee \psi)=P_{n}(\phi)+P_{n}(\psi)-P_{n}(\phi \wedge \psi)$.
d) Since $\phi \rightarrow \psi$ is a tautology it is the case that if $M \models_{\mu} \phi$ then $M \models_{\mu} \psi$. This implies

$$
\begin{equation*}
\left\{M \in \mathscr{M}_{n} \mid M \models_{\mu} \phi\right\} \subseteq\left\{M \in \mathscr{M}_{n}|M|=_{\mu} \psi\right\} . \tag{3.22}
\end{equation*}
$$

From Lemma 2.2.2 it follows that $P_{n}(\phi) \leq P_{n}(\psi)$.

Definition 3.1.7. An exclusive existential (universal) quantifier $\exists x / x_{1}, \cdots, x_{n}\left(\forall x / x_{1}, \cdots, x_{n}\right)$, read as there exists an (for all) $x$ other than $x_{1}, \cdots, x_{n}$, is defined by:

- $\exists x / x_{1}, \cdots, x_{l} \phi=\exists x\left(x \neq x_{1} \wedge \cdots \wedge x \neq x_{l} \wedge \phi\right)$.
- $\forall x / x_{1}, \cdots, x_{l} \phi=\forall x\left(x \neq x_{1} \wedge \cdots \wedge x \neq x_{l} \rightarrow \phi\right)$.

Remark 3.1.8. The list of variables in the exclusive quantifiers can be empty. This means that $\exists x / \varnothing p(x)$ is also an exclusive quantifier.

Lemma 3.1.9. Let $\circ$ be either $\vee$ or $\wedge$ and $x \notin \operatorname{fv}(\psi)$, then:
a) $\forall x \phi\left(x, z_{1}, \cdots, z_{l}\right) \equiv \forall x / y_{1}, \cdots, y_{n} \phi\left(x, z_{1}, \cdots, z_{l}\right) \wedge \bigwedge_{i=1}^{n} \phi\left(y_{i}, z_{1}, \cdots, z_{l}\right)$.
b) $\exists x \phi\left(x, z_{1}, \cdots, z_{l}\right) \equiv \exists x / y_{1}, \cdots, y_{n} \phi\left(x, z_{1}, \cdots, z_{l}\right) \vee \bigvee_{i=1}^{n} \phi\left(y_{i}, z_{1}, \cdots, z_{l}\right)$.
c) $\neg \forall x / y_{1}, \cdots, y_{n} \phi \equiv \exists x / y_{1}, \cdots, y_{n} \neg \phi$.
d) $\neg \exists x / y_{1}, \cdots, y_{n} \phi \equiv \forall x / y_{1}, \cdots, y_{n} \neg \phi$.
e) $\forall x / y_{1}, \cdots, y_{n}(\phi \circ \psi) \equiv\left(\forall x / y_{1}, \cdots, y_{n} \phi\right) \circ \psi$.
f) $\exists x / y_{1}, \cdots, y_{n}(\phi \circ \psi) \equiv\left(\exists x / y_{1}, \cdots, y_{n} \phi\right) \circ \psi$.
g) $\exists x / y_{1}, \cdots, y_{n}\left(\phi \vee \psi^{\prime}\right) \equiv\left(\exists x / y_{1}, \cdots, y_{n} \phi\right) \vee\left(\exists x / y_{1}, \cdots, y_{n} \psi^{\prime}\right)$.

Proof. Let $M$ be some model, then:
a) $M \models_{\mu} \forall x \phi\left(x, z_{1}, \cdots, z_{l}\right)$ if and only if for all $x M=_{\mu} \phi\left(x, z_{1}, \cdots, z_{l}\right)$ i.e. $M \models_{\mu} \phi\left(x, z_{1}, \cdots, z_{l}\right)$ for all $x$ except $y_{1}, \cdots, y_{l}$ and $M \models_{\mu} \phi\left(x, z_{1}, \cdots, z_{l}\right)$ for $y_{1}, \cdots, y_{l}$. This means $M \models_{\mu} \forall x / y_{1}, \cdots, y_{n} \phi\left(x, z_{1}, \cdots, z_{l}\right) \wedge$ $\bigwedge_{i=1}^{n} \phi\left(y_{i}, z_{1}, \cdots, z_{l}\right)$.
b) The proof is similar to the proof of Case a given above.
c) If $M \neq_{\mu} \neg \forall x / y_{1}, \cdots, y_{n} \phi$ then $M \not \vDash_{\mu} \forall x / y_{1}, \cdots, y_{n} \phi$ which means that there exists an $x$ unequal to $y_{1}, \cdots, y_{n}$ such that $M \not \models_{\mu} \phi$. This means that there exists an $x$ unequal to $y_{1}, \cdots, y_{n}$ such that $M \models_{\mu} \neg \phi$ i.e. $M \models_{\mu} \exists x / y_{1}, \cdots, y_{n} \neg \phi$.
d) The proof is similar to the proof of Case c given above.
e) Since $x$ is not a free variable in $\psi$ binding it with a quantifier will do nothing. From this fact the statement follows.
f) The proof is similar to the proof of Case e given above.
g) This follows from $\exists x\left(\phi \vee \psi^{\prime}\right)=(\exists x \phi) \vee\left(\exists x \psi^{\prime}\right)$ and Definition 3.1.7

Definition 3.1.10. $A \Gamma$-formula $\phi$ is a formula which contains no conventional universal and no existential quantifiers. This means $\phi$ can still contain exclusive quantifiers. The excluded variables must contain all free variables of the formula the quantifier is being applied to.

Definition 3.1.11. A free elementary part in a formula $\phi$ is an atomic formula in which all the variables are free.

Example 3.1.12. In the formula $\forall x(p(x, z) \wedge q(z)), q(z)$ is a free elementary part and $p(x, z)$ is not a free elementary part.

Lemma 3.1.13. For every formula $\phi$ there exists a $\Gamma$-formula $\psi$ such that $\phi \equiv \psi$.

Proof. Let $\phi(x)=\Pi_{l} x_{l} \Pi_{l-1} x_{l-1} \cdots \Pi_{1} x_{1} \psi$ be in prenex normal form. Using Lemma 3.1.9we construct a $\Gamma$-formula $\psi_{l}$ by iteratively excluding all free variables from the quantifiers. This algorithm can be seen in the following equations, where $\circ$ is given by $\wedge$ if $\Pi=\forall$ and $\vee$ if $\Pi=\exists$.

$$
\begin{gather*}
\psi_{i}=\Pi x_{i} /\left(\mathrm{fv}\left(\psi_{i-1}\right) \backslash\left\{x_{i}\right\}\right) \psi_{i-1} \circ\left(\circ_{z \in\left(\mathrm{fv}\left(\psi_{i-1}\right) \backslash\left\{x_{i}\right\}\right)} \psi_{i-1}\left[z \mapsto x_{i}\right]\right)  \tag{3.23}\\
\psi_{0}=\psi \tag{3.24}
\end{gather*}
$$

Example 3.1.14. Applying Lemma 3.1.13 results in:

$$
\begin{align*}
& \forall x \exists y((p(x, y, z) \wedge q(x)) \vee p(x, x, z))  \tag{3.25}\\
\equiv & \forall x(((\exists y / z, x(p(x, y, z) \wedge q(x))) \vee(p(x, z, z) \wedge q(x)) \vee(p(x, x, z) \wedge q(x))) \vee p(x, x, z))  \tag{3.26}\\
\equiv & (\forall x / z((\exists y / z, x(p(x, y, z) \wedge q(x))) \vee(p(x, z, z) \wedge q(x)) \vee(p(x, x, z) \wedge q(x))) \vee p(x, x, z)) \wedge  \tag{3.27}\\
& (((\exists y / z, x(p(z, y, z) \wedge q(z))) \vee(p(z, z, z) \wedge q(z)) \vee(p(z, z, z) \wedge q(z))) \vee p(z, z, z)) . \tag{3.28}
\end{align*}
$$

Example 3.1.15. Let $\Phi$ be as defined in 2.1.27, then an equivalent $\Gamma$-formula to the $1+1$ extensionality axiom $\psi$ can be found as follows:

$$
\begin{align*}
\psi & =\forall x_{1}\left(\bigwedge_{1 \leq i<j \leq 1} x_{i} \neq x_{j} \rightarrow \exists x_{1+1}\left(\bigwedge_{1 \leq i \leq 1} x_{i} \neq x_{1+1} \wedge \bigwedge_{\phi \in \Phi} \phi \wedge \bigwedge_{\phi \in \Phi^{C}} \neg \phi\right)\right)  \tag{3.29}\\
& \equiv \forall x_{1}\left(\exists x_{2}\left(x_{1} \neq x_{2} \wedge \bigwedge_{\phi \in \Phi} \phi \wedge \bigwedge_{\phi \in \Phi^{C}} \neg \phi\right)\right)  \tag{3.30}\\
& \equiv \forall x_{1}\left(\left(\exists x_{2} / x_{1}\left(\bigwedge_{\phi \in \Phi} \phi \wedge \bigwedge_{\phi \in \Phi^{C}} \neg \phi\right)\right)\right.  \tag{3.31}\\
& \left.\vee\left(x_{1} \neq x_{1} \wedge \bigwedge_{\phi \in \Phi} \phi\left[x_{1} \mapsto x_{2}\right] \wedge \bigwedge_{\phi \in \Phi^{C}} \neg \phi\left[x_{1} \mapsto x_{2}\right]\right)\right)  \tag{3.32}\\
& \equiv \forall x_{1} / \emptyset\left(\exists x_{2} / x_{1}\left(\bigwedge_{\phi \in \Phi} \phi \wedge \bigwedge_{\phi \in \Phi^{C}} \neg \phi\right)\right) \tag{3.33}
\end{align*}
$$

Lemma 3.1.16. Every $\Gamma$-formula $\phi$ can be represented as $\bigvee_{i \in I} \phi_{i} \wedge \psi_{i}$ for some index set $I$, where $\phi_{i}$ contains only free elementary parts and $\psi_{i}$ contains no free elementary parts.

Proof. Let $\phi=\Pi x / x_{1}, \cdots, x_{n} \psi$ be a $\Gamma$-formula which contains some free elementary part and some non free elementary part. If this is not the case, then we can apply the properties presented in Lemma 3.1.9 to rewrite $\phi$ into the desired form. Without loss of generality we can suppose that $\Pi$ is $\exists$, since $\forall x \psi \equiv \neg \exists x \neg \psi$. Because if $\exists x \neg \psi$ is in the desired form $\bigvee_{i \in I} \phi_{i} \wedge \psi_{i}$, then $\forall x \psi \equiv \neg \exists x \neg \psi \equiv \bigwedge_{i \in I} \neg \phi_{i} \vee \neg \psi_{i}$. This formula can then be transformed into its dnf using $\neg \phi_{i}$ and $\neg \psi_{i}$ as the variables for expansion, which does not change the range of the quantifiers and produces the desired form of $\forall x \psi$.

Let $\psi^{\prime}=\bigvee_{i \in I} \bigwedge_{j \in J_{i}} \psi_{i j}$ be the dnf of $\psi$ then using Lemma 3.1.9

$$
\begin{align*}
& \exists x / x_{1}, \cdots, x_{n} \psi \equiv \exists x / x_{1}, \cdots, x_{n} \psi^{\prime}  \tag{3.34}\\
= & \exists x / x_{1}, \cdots, x_{n}\left(\bigvee_{i \in I} \bigwedge_{I \in J_{i}} \psi_{i j}\right) \equiv \bigvee_{i \in I}\left(\exists x / x_{1}, \cdots, x_{n} \bigwedge_{j \in J_{i}} \psi_{i j}\right) . \tag{3.35}
\end{align*}
$$

Furthermore using Lemma 3.1.9 we can transform $\bigvee_{i \in I}\left(\exists x / x_{1}, \cdots, x_{n} \bigwedge_{j \in J_{i}} \psi_{i j}\right)$ into the desired form. This procedure can easily be generalized for $\phi=\Pi x_{1} / x_{11}, \cdots, x_{1 n} \cdots \Pi x_{m} / x_{m 1}, \cdots, x_{m n} \psi$. Since there are only finitely many quantifiers in $\phi$ and the free elementary parts are moved up one quantifier each time this procedure is applied. This means that at some point all free elementary parts will have been removed from the range of all quantifiers.

Definition 3.1.17. Let $\phi$ be a formula and $P_{n}$ be as in Definition 3.1.1 Then $\phi$ is called 0-admissible if for every $k \in \mathbb{N}$

$$
\begin{equation*}
\lim _{n \rightarrow \infty} n^{k} P_{n}(\phi)=0 \tag{3.36}
\end{equation*}
$$

$\phi$ is called 1-admissible if for every $k \in \mathbb{N}$

$$
\begin{equation*}
\lim _{n \rightarrow \infty} n^{k}\left(1-P_{n}(\phi)\right)=0 \tag{3.37}
\end{equation*}
$$

If $\phi$ is 0 -admissible or 1-admissible it is called admissible.
Lemma 3.1.18. Let $\phi$ and $\psi$ be $\Gamma$-formulas with no constant symbols nor function symbols and $P_{n}$ be as in Definition 3.1.1 Suppose that $\phi$ and $\psi$ are admissible. Then:
a) $\neg \phi$ is admissible.
b) $\phi \vee \psi$ is admissible.

Proof. a) This follows by using Lemma 2.2.2 since $\phi$ is admissible.
b) Since $P_{n}(\phi), P_{n}(\psi) \leq P_{n}(\phi \vee \psi) \leq P_{n}(\phi)+P_{n}(\psi)$ and $\phi$ and $\psi$ are admissible, then $\phi \vee \psi$ is admissible.

Lemma 3.1.19. Let $\phi=\Pi x / y_{1}, \cdots, y_{l} \psi$ where $\psi$ does not contain quantifiers, free elementary parts, constant symbols nor function symbols. Then it is the case that $\phi$ is admissible.

Proof. Since we can reduce $\exists$ to $\forall$ with negation, we only consider the case $\phi=\forall x / y_{1}, \cdots, y_{l} \psi$. Let $P_{n}$ be as in Definition 3.1.1

Let $\psi_{1}=\sigma_{1} \vee \sigma_{2}$ be the ddnf of $\psi$ according to Lemma 2.1.23. Using Lemma 3.1.6 it follows that $P_{n}\left(\sigma_{1} \vee \sigma_{2}\right)=P_{n}\left(\sigma_{1}\right)+P_{n}\left(\sigma_{2}\right)-P_{n}\left(\sigma_{1} \wedge \sigma_{2}\right)=P_{n}\left(\sigma_{1}\right)+P_{n}\left(\sigma_{2}\right)$ since $\psi_{1}$ is in ddnf. This fact combined with Definition 3.1.1 and Lemma 3.1.6 allows us to calculate $P_{n}\left(\psi_{1}\right)$, which is important in order to obtain a constructive procedure. From $\psi_{1} \equiv \psi$ it follows that $P_{n}\left(\psi_{1}\right)=P_{n}(\psi)$.

From Definition 3.1 .9 for a Model size of $n$, with $\mu\left(x_{t}\right)=t$, where $x_{t}$ are new variables, it follows that:

$$
\begin{equation*}
M \models_{\mu} \forall x / y_{1}, \cdots, y_{l} \psi_{1} \leftrightarrow \bigwedge_{t=1, x \neq y_{1}, \cdots, y_{l}}^{n} \psi_{1}\left[x \mapsto x_{t}\right] \text { for all models } M \in \mathscr{M}_{n} \tag{3.38}
\end{equation*}
$$

Since $\psi_{1}\left[x \mapsto x_{t}\right]$ is independent of $\psi_{1}\left[x \mapsto x_{t^{\prime}}\right]$ for $t \neq t^{\prime}$ given $\mu$, because $\phi$ contains no free elementary parts, it follows that:

$$
\begin{equation*}
P_{n}(\phi)=P_{n}\left(\bigwedge_{t=1, x \neq y_{1}, \cdots, y_{l}}^{n} \psi_{1}\left[x \mapsto x_{t}\right], \mu\right)=P_{n}(\psi)^{n-l} \tag{3.39}
\end{equation*}
$$

Given the above, if $P_{n}(\psi)=1$, then $P(\psi) \rightarrow^{n \rightarrow \infty} 1$ exponentially fast. If $P(\psi) \neq 1$, it follows that $P(\phi)<1$ which implies $P(\phi) \rightarrow^{n \rightarrow \infty} 0$ exponentially fast. From both these facts it follows that $\phi$ is admissible.

Theorem 3.1.20. Let $P_{n}$ be as in Definition 3.1.1 Then every $\Gamma$-formula $\phi$ with no free elementary parts, constant symbols nor function symbols is admissible. This implies that $\lim _{n \rightarrow \infty} P_{n}(\phi) \in\{0,1\}$.
Proof. We use induction on the number of exclusive quantifiers $m$ of the formula. This number must be bigger than zero, since the formula contains no free elementary parts.
Base case ( $m=1$ ): This follows from Lemma 3.1.19.
Induction case: Suppose the theorem is valid for all formulas with less than $m+1$ quantifiers. We now prove that this is also the case for all formulas $\phi$ with $m+1$ quantifiers. Suppose $\phi=\psi_{1} \circ \psi_{2}$ where $\circ$ is one of $\vee, \wedge, \leftrightarrow, \rightarrow$. Then the induction case follows from Lemma 3.1.18 and the induction hypothesis.
Suppose that $\phi=\Pi x / y_{1}, \cdots, y_{l} \psi$ where $\psi$ contains at most $m$ quantifiers. Assume that $\Pi=\exists$, since the $\Pi=\forall$ case follows by applying Lemma's 2.2.2 and 3.1.9.
From Definition 3.1 .9 for a Model size of $n$ it follows that $\exists x / y_{1}, \cdots, y_{l} \phi \equiv \bigvee_{x=1, x \neq y_{1}, \cdots y_{l}}^{n} \phi$. From Lemma 2.2.2 if follows that:

$$
\begin{equation*}
P_{n}(\phi) \leq P_{n}\left(\exists x / y_{1}, \cdots, y_{l} \phi(x)\right) \leq n P_{n}(\phi) . \tag{3.40}
\end{equation*}
$$

We continue with a proof by cases.
Case 1: $\phi$ is 0-admissible. Using equation 3.40 and the fact that $\phi$ is 0 -admissible, it follows for all $k \in \mathbb{N}$ that:

$$
\begin{equation*}
0 \leq \lim _{n \rightarrow \infty} n^{k} P_{n}\left(\exists x / y_{1}, \cdots, y_{l} \phi(x)\right) \leq \lim _{n \rightarrow \infty} n^{k+1} P_{n}(\phi)=0 \tag{3.41}
\end{equation*}
$$

From this it follows that $\exists x / y_{1}, \cdots, y_{l} \phi(x)$ is 0 -admissible.
Case 2: $\phi$ is 1-admissible. Using equation 3.40 and the fact that $\phi$ is 1 -admissible, it follows for all $k \in \mathbb{N}$ that:

$$
\begin{equation*}
0=\lim _{n \rightarrow \infty} n^{k}\left(1-P_{n}(\phi)\right) \geq \lim _{n \rightarrow \infty} n^{k}\left(1-P_{n}\left(\exists x / y_{1}, \cdots, y_{l} \phi(x)\right)\right) \geq 0 \tag{3.42}
\end{equation*}
$$

From this it follows that $\exists x / y_{1}, \cdots, y_{l} \phi(x)$ is 1 -admissible.
From this it follows that $\exists x / y_{1}, \cdots, y_{l} \phi(x)$ is admissible.

Example 3.1.21. Let $P_{n}$ be as in Definition 3.1.1 and $p$ be a unary predicate symbol with $P_{n}(p(x))=$ $s \in(0,1)$ then:

$$
\begin{gather*}
\lim _{n \rightarrow \infty} P_{n}(\forall x p(x))=\lim _{n \rightarrow \infty} s^{n}=0 .  \tag{3.43}\\
\lim _{n \rightarrow \infty} P_{n}(\exists x p(x))=\lim _{n \rightarrow \infty} 1-(1-s)^{n}=1 . \tag{3.44}
\end{gather*}
$$

Example 3.1.22. Every extensionality axiom is 1-admissibl ${ }^{1}$ In example 3.1.15 we transformed a $1+1$ extensionality axiom into a $\Gamma$-formula. Now we will see that it is 1 -admissible. Let $\psi=$ $\wedge_{\phi \in \Phi} \phi \wedge \wedge_{\phi \in \Phi^{C}} \neg \phi$. From Lemma 3.1.19 it follows that:

$$
\begin{align*}
& \lim _{n \rightarrow \infty} n P_{n}\left(\forall x_{2} / x_{1}(\neg \psi)\right)  \tag{3.45}\\
= & \lim _{n \rightarrow \infty} n P_{n}\left(\wedge_{i=1, i \neq x_{1}}^{n}\left(\neg \psi\left[x_{2} \mapsto i\right]\right)\right)  \tag{3.46}\\
= & \lim _{n \rightarrow \infty} n P_{n}\left(\neg \psi\left[x_{2} \mapsto j\right]\right)^{n-1}=0 \tag{3.47}
\end{align*}
$$

[^2]for some $j \neq x_{1}$. From which it follows that:
\[

$$
\begin{align*}
& \lim _{n \rightarrow \infty} P_{n}\left(\forall x_{1} / \emptyset\left(\exists x_{2} / x_{1}(\psi)\right)\right)  \tag{3.48}\\
= & \lim _{n \rightarrow \infty} 1-P_{n}\left(\exists x_{1} / \emptyset \neg\left(\exists x_{2} / x_{1}(\psi)\right)\right)  \tag{3.49}\\
\geq & \lim _{n \rightarrow \infty} 1-n P_{n}\left(\neg\left(\exists x_{2} / x_{1}(\psi)\right)\right)  \tag{3.50}\\
= & \lim _{n \rightarrow \infty} 1-n P_{n}\left(\forall x_{2} / x_{1}(\neg \psi)\right)=1 \tag{3.51}
\end{align*}
$$
\]

Theorem 3.1.23. Let $P_{n}$ be as in Definition 3.1.1 then for every formula $\phi(x)$ with $k \in \mathbb{N}$ free variables $\left\{x_{1}, \cdots, x_{k}\right\}$ without constant nor function symbols, there exists a formula $\psi(x)$ with $k$ free variables and without quantifiers such that $\lim _{n \rightarrow \infty} P_{n}(\sigma)=1$ where

$$
\begin{equation*}
\sigma=\forall x_{1}, \cdots, x_{k}\left(\phi\left(x_{1}, \cdots, x_{k}\right) \leftrightarrow \psi\left(x_{1}, \cdots, x_{k}\right)\right) . \tag{3.52}
\end{equation*}
$$

Proof. By Lemma 3.1.13 we can transform $\phi$ into a $\Gamma$-formula and by Lemma 3.1.16 we ob$\operatorname{tain} \phi \equiv \bigvee_{i \in I} \psi_{i} \wedge \phi_{i}$ for some finite index set $I$, where $\psi_{i}$ contains only free elementary parts and $\phi_{i}$ contains no free elementary parts. Since $\phi_{i}$ is a $\Gamma$-formula with no free elementary parts, we can apply to each $\phi_{i}$ the 0-1-Law for $\Gamma$-formulas as given by Theorem 3.1.20 Let $J=\left\{i \mid \lim _{n \rightarrow \infty} P_{n}\left(\phi_{i}\right)=1\right\}$ and $\psi=\bigvee_{i \in J} \psi_{i}$.

The formula $\theta=\bigwedge_{i \in J} \phi_{i} \wedge \bigwedge_{i \in I \backslash J} \neg \phi_{i}$ implies $\forall x_{1}, \cdots, x_{k}\left(\phi\left(x_{1}, \cdots, x_{k}\right) \leftrightarrow \psi\left(x_{1}, \cdots, x_{k}\right)\right)$. Since for each model $M$ such that $M \models_{\mu} \theta$ it follows that $M \models_{\mu} \phi$ if and only if $M \models_{\mu} \psi$, because $\phi \equiv \bigvee_{i \in I} \psi_{i} \wedge \phi_{i}$ and $\psi=\bigvee_{i \in J} \psi_{i}$.

Lemma 3.1.6implies that

$$
\begin{equation*}
P_{n}(\theta) \leq P_{n}\left(\forall x_{1}, \cdots, x_{k}\left(\phi\left(x_{1}, \cdots, x_{k}\right) \leftrightarrow \psi\left(x_{1}, \cdots, x_{k}\right)\right)\right) . \tag{3.53}
\end{equation*}
$$

For all $i \in J \lim _{n \rightarrow \infty} P_{n}\left(\phi_{i}\right)=1$ and for all $i \in I \backslash J \lim _{n \rightarrow \infty} P_{n}\left(\phi_{i}\right)=0$. Using Lemma3.1.6 it follows that

$$
\begin{equation*}
P_{n}(\theta)=1-P_{n}(\neg \theta)=1-P_{n}\left(\bigvee_{i \in J} \neg \phi_{i} \vee \bigvee_{i \in I \backslash J} \phi_{i}\right) \geq 1-\sum_{i \in J} P_{n}\left(\neg \phi_{i}\right)-\sum_{i \in I \backslash J} P_{n}\left(\phi_{i}\right) \rightarrow^{n \rightarrow \infty} 1 \tag{3.54}
\end{equation*}
$$

This means that $\lim _{n \rightarrow \infty} P_{n}(\forall x .(\phi(x) \leftrightarrow \psi(x)))=1$.
Corollary 3.1.24. Let $P_{n}$ be as in Definition 3.1.1 For every formula $\phi(x)$ with $k \in \mathbb{N}$ free variables $\left\{x_{1}, \cdots, x_{k}\right\}$ without function symbols and which may contain constants, there exists a formula $\psi(x)$ with $k$ free Variables and without quantifiers such that $\lim _{n \rightarrow \infty} P_{n}(\sigma)=1$ where

$$
\begin{equation*}
\sigma=\forall x_{1}, \cdots, x_{k}\left(\phi\left(x_{1}, \cdots, x_{k}\right) \leftrightarrow \psi\left(x_{1}, \cdots, x_{k}\right)\right) \tag{3.55}
\end{equation*}
$$

Proof. Let $\phi_{0}$ be the result of replacing each constant $c_{i}$ in $\phi$ by a new free variable $z_{i}$. Using Theorem 3.1.23 we obtain a formula $\psi_{0}$ with the same free variable $\left\{x_{1}, \cdots, x_{k}, z_{1}, \cdots, z_{l}\right\}$ as $\phi_{0}$ such that $\lim _{n \rightarrow \infty} P_{n}\left(\theta_{0}\right)=1$ for $\theta_{0}=\forall x_{1}, \cdots, x_{k}\left(\phi_{0}\left(x_{1}, \cdots, x_{k}\right) \leftrightarrow \psi_{0}\left(x_{1}, \cdots, x_{k}\right)\right)$. Let $\psi$ be the result of replacing each free variable $z_{i}$ in $\psi_{0}$ by their corresponding constant $c_{i}$. Since some free variables are replaced by constants, $\theta_{0}$ implies $\theta=\forall x_{1}, \cdots, x_{k}\left(\phi\left(x_{1}, \cdots, x_{k}\right) \leftrightarrow\right.$ $\left.\psi\left(x_{1}, \cdots, x_{k}\right)\right)$. By Lemma 3.1.6 it follows that $P_{n}\left(\theta_{0}\right) \leq P_{n}(\theta) \leq 1$. Since $\lim _{n \mapsto \infty} P_{n}\left(\theta_{0}\right)=1$ it follows that $\lim _{n \mapsto \infty} P_{n}(\theta)=1$.

## CHAPTER 4

We have proven the quantifier elimination theorem for first order logic, when there are no function symbols, under a more general probability measure than the one used by Glebskii et. al. . To prove this theorem, we first proved that every $\Gamma$-formula is admissible, from which the 0-1 Law for $\Gamma$-formulas followed. Later we separated the formula $\phi$, from which the quantifiers should be eliminated, into free elementary parts and non free elementary parts. Using the fact that every $\Gamma$-formula is admissible we could remove non-free elementary parts, by making them equivalent to "true" or "false", when the model size goes to infinity.

This approach stands in contrast to the one presented by Fagin. Glebskii et. al. directly constructed a $\Gamma$-formula and could apply the 0-1 Law to it, as we saw in this thesis. Fagin's approach first proved that the set $T$ of extensionality axioms is consistent and complete i.e. a contradiction cannot be derived and every formula can be derived from a finite subset of them. Afterwards Fagin proves that every extensionality axiom is 1 -admissible. Finally to determine if a formula is 1 -admissible or 0-admissible one needs to prove that the formula does or does not follow from the extensionality axioms.

Using the steps given in the proof one can reduce the calculation cost of predicting the probability that a formula is "true". Since eliminating the quantifiers from the formula, allows it to be evaluated independently of the model size, because only free variables and constants remain. This step can be thought of as a preprocessing or compilation step, which according to [8] is exponentially expensive. Even though this result only determines what happens when the model size goes to infinity, one could predict what happens at finite model sizes, using an error bound. Given a formula $\phi$ we can construct a formula $\psi$ without quantifiers such that $\lim _{n \rightarrow \infty} P_{n}\left(\forall x_{1}, \cdots, x_{k}(\phi \leftrightarrow \psi)\right)=1$ as given by Theorem 3.1.23 This means we can approximate the probability that a model fulfills $\phi$ by calculating the probability that a model fulfills $\psi$. We saw in Lemma 3.1 .19 that the convergence rate is exponential in the model size, which means that the error when approximating the probability that a model fulfills $\phi$ is small. While this provides a good way to generate the approximation further work is still needed in order to implement this as a practical algorithm and to determine if it can applied in practice for statistical relational AI.

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[^0]:    ${ }^{1}$ This is a very simple characterisation of the problem and will be further formalized in the following sections.

[^1]:    ${ }^{1}$ Constants are not considered for most results in this work, except for Corollary 3.1.24

[^2]:    ${ }^{1}$ This is one of the key results in Fagin's proof of the 0-1 Law.

