Knowledge Representation and Reasoning

Allen's Interval Calculus
An Example for Constraint Processing

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This lecture

- Motivation
- Intervals and relations between them
- Composition Table
- Path-Consistency
- Incompleteness of Path-Consistency
- NP-Completeness
Temporal Notions can be

• *precise* (e.g. dates like 20.1.2015, 13:15)

• *imprecise*

Examples:
  - Natural Language:
    "during my holidays", "after the lecture", ...
  - Planning:
    we don't want to commit to exact time points
  - Scenario Descriptions:
    we do not have exact time points
    "first this", "then that"

Nevertheless we do reason about temporal notions:
"If A before B and B before C then A is before C."
We want to model the temporal relations between events

Two Possibilities

- **Time Points** (Variables denoting time points)
  instantaneous events,
  start and end-time of events.

- **Time Intervals** (Variables denoting time intervals)
  Events have a duration.
  In this lecture we focus on this case.
Example

Scenario for multimedia generation:

P1: Display Picture 1
P2: play audio "shut off the device"
P3: play audio "remove the plug"
P4: point to plug in Picture 1

Temporal relations between events:

P2 should happen during P1
P3 should happen during P1
P2 should happen before or directly precede P3
P4 should happen during or start with P3

⇒ P4 happens after P2
Wanted

- **Check for inconsistencies**

  A after B, B after C, C after A is inconsistent

- **Infer implicit relations**

  A after B, B after C $\Rightarrow$ A after C
13 Base Relations Between Intervals

X before Y
X meets Y
X overlaps Y

X during Y
X starts Y
X finishes Y
X equals Y

Shortcuts:
before: b or <
after: a or >
meets: m
overlaps: o
starts: s
finishes: f
equals: e, ≡
Observation

The 13 relations are

- **Mutually disjoint**
  if (X r Y) for any of the relations then
  not (X s Y) for every other relation

- **Complete:**
  for two intervals X, Y exactly one of the relations X r Y holds.
Sets of Relations

In many cases we do not exactly know the relations

But we might know:

\[ X \circ Y \text{ or } X \cap Y \] (X overlaps Y, or X meets Y)

Shortcut: \( X \{\circ, \cap\} Y \)

This way we get \( 2^{13} \) imprecise relations
(including \( \emptyset \) (empty relation) and
\( \top \) (universal relation, also written \( B \))

Example of a qualitative description

\[ \{X \{\circ, \cap\} Y, Y \{\cap\} Z, X \{\circ, \cap\} Z\} \]
The Previous Example

P2 should happen during P1
P3 should happen during P1
P2 should happen before or directly precede P3
P4 should happen during or start with P3

Constraint Graph:

\[
\begin{array}{c}
\text{P2} & \text{d} & \text{P1} \\
\text{b,m} & \text{?} & \text{d} \\
\text{P2} & \text{P3} & \text{P4} \\
\text{?} & \text{d,s} & \Rightarrow \text{P4} \{d\} \text{ P1}
\end{array}
\]
Path-Consistency Method

Until nothing changes any more:
• Choose $X \text{ r } Y$ and $Y \text{ s } Z$, with $X \text{ t } Z$
• Compute $t' = (r \circ s) \cap t$, this becomes the new $X \text{ t' } Z$
• If $t'$ becomes empty then the network is inconsistent

Problem:
What is $(r \circ s)$, for example $(\text{meets } \circ \text{ during})$?
We need $13 \times 13 = 269$ compositions.
Examples

before ◦ before = before

\[
\begin{array}{ccc}
X & Y & Z \\
\hline
\end{array}
\]

meets ◦ overlaps = before

\[
\begin{array}{ccc}
X & Y & Z \\
\hline
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\end{array}
\]

167 more combinations need to be determined
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Path-Consistency Method with the Composition Table

Until nothing changes any more:

- Choose $X \mathbin{r} Y$ and $Y \mathbin{s} Z$, with $X \mathbin{t} Z$
- Compute $t' = (r \circ s) \cap t$, this becomes the new $X \mathbin{t'} Z$
- If $t'$ becomes empty then the network is inconsistent

**Example:** $(r \circ s) = \{d,s\} \circ \{o,m\}$

From the composition table we get:

<table>
<thead>
<tr>
<th>Operation</th>
<th>Result</th>
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<tr>
<td>$d \circ o$</td>
<td>${b,o,m,d,s}$</td>
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<tr>
<td>$d \circ m$</td>
<td>${b}$</td>
</tr>
<tr>
<td>$s \circ o$</td>
<td>${b,o,m}$</td>
</tr>
<tr>
<td>$s \circ m$</td>
<td>${b}$</td>
</tr>
</tbody>
</table>

Thus, $\{d,s\} \circ \{o,m\} = \{b,o,m,d,s\}$
The Previous Example Again

Constraint Graph:

We get

\[ P_4 \rel r P_1 \text{ where } r = (\{d,s\} \circ \{d\}) \cap \top = \{d\} \]

\[ ((r \cap \top) = r, \text{ for all relations } r) \]

We get

\[ P_4 \rel s P_2 \text{ where } s = (\{d,s\} \circ \{b^{-1}, m^{-1}\}) \cap \top = \{b^{-1}, m^{-1}\} \]
This network is inconsistent, but path-consistency does not discover it.

D s A and D s C as well as
D m A and D m C cause A \{s,s^{-1}\} C
which contradicts A \{f,f^{-1}\} C

D s A causes B d A
D m C causes C d B
which contradicts A \{f,f^{-1}\} C
Theorem (Kautz & Vilain)

Constraint Satisfaction is NP-hard for Allen's Interval Calculus

Proof: Reduction of 3-colorability of graphs to Interval Calculus

Let $G = (V,E)$, $V = \{v_1, ..., v_n\}$ be a graph to be 3-colored.

We use intervals $\{v_1, ..., v_n, r, g, b\}$ with the following constraints:

\[
\begin{align*}
  r &\quad \{m\} & g \\
  g &\quad \{m\} & b \\
  v_i &\quad \{m, \equiv, m^{-1}\} & g \quad \text{for all } v_i \\
  v_i &\quad \{m, m^{-1}, b, a\} & v_j \quad \text{for all } (v_i, v_j) \in E
\end{align*}
\]

The constraint system is satisfiable iff the graph is 3-colorable.
Summary

Allen's Interval Calculus

• 13 basic relations between intervals
• Constraint System
• Composition Table
• Path Consistency
• Incompleteness of Path Consistency
  Special cases have been found, where path consistency is complete
• NP-hardness (NP-completeness is then obvious)