Exercise 5-1  Loopcheck

a) Give a precise definition of the $\rightarrow_3$-rule with loopcheck.

**Proposed Solution:**

$S \rightarrow_3$ loopcheck: $\{xry\} \cup S$, if

1. $\rightarrow_\forall$, $\rightarrow_\lor$, $\rightarrow_\land$, $\rightarrow_\subseteq$ are not applicable
2. $\forall x: \exists r \varphi^x \in S$
3. $\forall y: \varphi^y \in S$
4. $\{\psi \mid x: \forall r \psi^x \in S\} \subseteq \{\psi \mid y: \psi^y \in S\}$

b) Check the consistency of $A$ given the axiom $A \subseteq \exists r \top \land \forall r (A \land B)$

**Proposed Solution:**

T-Box: $A \subseteq \exists r \top \land \forall r (A \land B)$

$x: A$

$x: A \subseteq \exists r \top \land \forall r (A \land B)$

$\mid \rightarrow_\subseteq$

$x: \exists r \top \land \forall r (A \land B)$

$\mid \rightarrow_\land$

$x: \exists r \top$

$x: \forall r (A \land B)$

$\mid \rightarrow_3$

$xry$

$y: \top$

$y: A \subseteq \exists r \top \land \forall r (A \land B)$

$\mid \rightarrow_\psi$

$y: A \land B$

$\mid \rightarrow_\land$

$y: A$

$y: B$

$\mid \rightarrow_\subseteq, \rightarrow_\land$

$y: \exists r \top$

$y: \forall r (A \land B)$

$\mid \rightarrow_3$ loopcheck

$yry$

$\mid$

open
c) Check the consistency of Node △ Edge given the axioms

- Edge ⊑ ∃ touches Node
- Node ⊑ ∀ touches Edge

**Proposed Solution:**

\[ x : \text{Node} \triangle \text{Edge} \]
\[ x : \text{Node} \subseteq \forall \text{touches Edge} \]
\[ x : \text{Node} \subseteq \exists \text{touches Node} \]
\[ x : \text{Node} \]
\[ x : \text{Edge} \]
\[ x : \forall \text{touches Edge} \]
\[ x : \exists \text{.touches Node} \]
\[ \xrightarrow{\exists} \text{loopcheck} \]
\[ x \text{ touches } x \]

**Exercise 5-2 Qualified Number Restrictions: Bon Appetit!**

Consider the concepts Food, Spice and the relation contains. In addition we have the following information.

- Mensa-Food is a Food not containing more than 2 Spices.
- Curry is a Mensa-Food with at least 5 Spices.

a) Formulate the above information as \(\mathcal{ALC}\)-axioms with qualified number restrictions.

**Proposed Solution:**

\[ \text{Mensa-Food} = \text{Food} \sqcap \text{atmost 2 contains Spice} \]
\[ \text{Curry} = \text{Mensa-Food} \sqcap \text{atleast 5 contains Spice} \]
b) Check with the Tableau Calculus the consistency of the concept Curry with the axioms.

**Proposed Solution:**

Curry = Mensa-Food ∩ atleast 5 contains Spice

= Food ∩ atmost 2 contains Spice ∩ atleast 5 contains Spice

\[ x : \text{Food} \cap \text{atmost 2 contains Spice} \cap \text{atleast 5 contains Spice} \]

\[ \rightarrow_{\gamma} \]

\[ x : \text{Food} \]

\[ x : \text{atmost 2 contains Spice} \ (\ast) \]

\[ x : \text{atleast 5 contains Spice} \ (\ast\ast) \]

\[ \rightarrow_{\geq \ast} (\text{mit } \ast\ast) \]

\[ x \text{ contains } y_{1} \]

\[ y_{1} : \text{Spice} \]

\[ y_{1} : \text{Spice} \]

\[ \perp \rightarrow_{\perp} \]

\[ x \text{ contains } y_{2} \]

\[ y_{2} : \text{Spice} \]

\[ y_{2} : \text{Spice} \]

\[ \perp \rightarrow_{\perp} \]

\[ x \text{ contains } y_{3} \]

\[ y_{3} : \text{Spice} \]

\[ y_{3} : \text{Spice} \]

\[ \perp \rightarrow_{\leq} \]

\[ \perp \rightarrow_{\perp} \]

Also: Curry is inconsistent.

---

**Exercise 5-3   Negation Normalform with Number Restrictions**

The NNF of unqualified number restrictions is defined by the following rules:

- \( \neg (\text{atleast } n \ r) \rightarrow \text{atmost } (n - 1) \ r \)
- \( \neg (\text{atmost } n \ r) \rightarrow \text{atleast } (n + 1) \ r \)

Prove this using the semantics of the operators.

**Proposed Solution:**

Let \( \mathcal{I} \) be an arbitrary interpretation.

\[ [\neg (\text{atleast } n \ r)]^{\mathcal{I}} = \]

\[ \mathcal{I} \setminus (\text{atleast } n \ r)^{\mathcal{I}} = \]

\[ \mathcal{I} \setminus \{x \in \mathcal{I} \mid |\{y \mid (x, y) \in r^{\mathcal{I}}\}| \geq n\} = \]

\[ \{x \in \mathcal{I} \mid \text{not } |\{y \mid (x, y) \in r^{\mathcal{I}}\}| \geq n\} = \]
\{x \in \Im_D \mid \{y \mid (x,y) \in r^3}\} \leq n - 1 = \\
[\text{atmost}(n - 1) \ r]^3

\text{the other case is analogous.}

Exercise 5-4  \textit{ALC with Number Restrictions}

Consider the following facts:

- Farmer Krause owns atleast 2 black horses.
- Farmer Krause owns atleast 2 white horses.
- Farmer Krause owns atmost 3 horses (with some colour)

What is the consequence for zebras?

Try to model the facts (including the consequences for zebras) as \textit{ALC}-formulae (with number restrictions).

Proposed Solution:

1. Krause : \textit{atleast} 2 owns (Horse \sqcap \exists \text{has-Colour black})
2. Krause : \textit{atleast} 2 owns (Horse \sqcap \exists \text{has-Colour white})
3. Krause : \textit{atmost} 3 owns Horse
4. Krause : owns \textit{P} _1
5. \textit{P} _1 : \text{Horse} \quad (1 \Rightarrow \textit{atleast})
6. \textit{P} _1 : \exists \text{has-Colour black} \quad (1 \Rightarrow \textit{atleast})
7. Krause : owns \textit{P} _2
8. \textit{P} _2 : \text{Horse} \quad (1 \Rightarrow \textit{atleast})
9. \textit{P} _2 : \exists \text{has-Colour black} \quad (1 \Rightarrow \textit{atleast})
10. Krause : owns \textit{P} _3
11. \textit{P} _3 : \text{Horse} \quad (2 \Rightarrow \textit{atleast})
12. \textit{P} _3 : \exists \text{has-Colour white} \quad (2 \Rightarrow \textit{atleast})
13. Krause : owns \textit{P} _4
14. \textit{P} _4 : \text{Horse} \quad (2 \Rightarrow \textit{atleast})
15. \textit{P} _4 : \exists \text{has-Colour white} \quad (2 \Rightarrow \textit{atleast})

Hence there are 4 horses. We have to record the Unique-Name Assumption for \textit{P} _1 and \textit{P} _2, as well as for \textit{P} _3 and \textit{P} _4. Because of 3 (Krause has only 3 horses), we must equate 2 horses.

There are the following possibilities: \textit{P} _1 = \textit{P} _3, \textit{P} _1 = \textit{P} _4, \textit{P} _2 = \textit{P} _3, \textit{P} _2 = \textit{P} _4.

We start with \textit{P} _1 = \textit{P} _3. With this we get: 12': \textit{P} _1 : \exists \text{has-Colour white}. \textit{P} _1 is therefore a horse, which is black (6) and white (12'), a Zebra. We could expand the existential quantifier further, but don’t find a contradiction. The branch remains open. From the canonical model we obtain that farmer Krause must have a zebra.

The other three possibilities \textit{P} _1 = \textit{P} _4, \textit{P} _2 = \textit{P} _3, \textit{P} _2 = \textit{P} _4 yield the same result.
Exercise 5-5 Repetition: Tableau with Qualified Number Restrictions

Use the Tableau Calculus to check the consistency of:

a) atmost 2 \( r \top \sqcap \) at least 3 \( r \psi \)

b) atmost 2 has-Pet \( \top \sqcap \exists \) has-Pet Horse \( \sqcap \exists \) has-Pet Cat \( \sqcap \exists \) has-Pet Dog
   with the T-Box axioms: \( \neg (\text{Cat} \sqcap \text{Dog}), \neg (\text{Horse} \sqcap \text{Cat}), \neg (\text{Horse} \sqcap \text{Dog}) \).

Proposed Solution:

a) We use the optimized method and shorten the proof a bit.

1. \( x : \text{atmost 2 } r \top \sqcap \text{atleast 3 } r \psi \)
2. \( x : \text{atmost 2 } r \top \quad (1 : \mapsto_\top) \)
3. \( x : \text{atleast 3 } r \psi \)
4. \( x r y_1 \quad (3 : \mapsto_\text{atleast}) \)
5. \( y_1 : \psi \)
6. \( x r y_2 \quad (3 : \mapsto_\text{atleast}) \)
7. \( y_2 : \psi \)
8. \( x r y_3 \quad (3 : \mapsto_\text{atleast}) \)
9. \( y_3 : \psi \)
10. Contradiction with 2 and unique name assumption \( y_1, y_2, y_3 \)

According to the optimized method one has to make a case distinction each time after 5, 7 and 9: \( y_i : \top \) und \( y_i : \neg \top \). The first case is always true and the second case always false. Therefore one can skip this.

b) The axioms in NNF are:

\( Ax_1 : \neg \text{Cat} \sqcup \neg \text{Dog} \quad \text{alternative: } \text{Cat} \sqsubseteq \neg \text{Dog} \quad \text{and } \text{Dog} \sqsubseteq \neg \text{Cat} \)

\( Ax_2 : \neg \text{Horse} \sqcup \neg \text{Cat} \quad \text{alternative: } \text{Horse} \sqsubseteq \neg \text{Cat} \quad \text{and } \text{Cat} \sqsubseteq \neg \text{Horse} \)

\( Ax_3 : \neg \text{Horse} \sqcup \neg \text{Dog} \quad \text{alternative: } \text{Horse} \sqsubseteq \neg \text{Dog} \quad \text{and } \text{Dog} \sqsubseteq \neg \text{Horse} \)

Tableaux:

1. \( x : \text{atmost 2 } \text{has-Pet} \)
2. \( x : \exists \text{has-Pet Horse} \)
3. \( x : \exists \text{has-Pet Cat} \)
4. \( x : \exists \text{has-Pet Dog} \)
5. \( x \text{ has-Pet } p \quad (2 : \mapsto_\exists) \)
6. \( p : \text{Horse} \)
7. \( p : \neg \text{Cat} \quad (6 : Ax_2) \)
8. \( p : \neg \text{Dog} \quad (6 : Ax_3) \)
9. \( x \text{ has-Pet } k \quad (3 : \mapsto_\exists) \)
10. \( k : \text{Cat} \)
11. \( k : \neg \text{Dog} \quad (10 : Ax_1) \)
12. \( k : \neg \text{Horse} \quad (10 : Ax_2) \)
13. \( x \text{ has-Pet } h \quad (4 : \mapsto_\exists) \)
14. \( h : \text{Dog} \)
15. \( h : \neg \text{Cat} \quad (10 : Ax_1) \)
16. \( h : \neg \text{Horse} \quad (10 : Ax_3) \)

At 7, 8 and 11, 12 and 15, 16 we have applied the corresponding axioms, did case distinction and closed one path immediately. For \( p, h, k \) we must apply the 3rd axiom and make a
case distinction This can, however, be deferred until we have found a contradiction, in which case the case distinction has become unnecessary.

\( x \) has now 3 pets, because of 1. he can have only 2 pets. Therefore we must equate 2 pets and consider the following cases: \( p = k, p = h \) und \( k = h \).

We follow the branch \( p = k \) replacing \( k \) by \( p \).

1. \( x : \text{atmost 2 has-Pet} \)
2. \( x : \exists \text{ has-Pet Horse} \)
3. \( x : \exists \text{ has-Pet Cat} \)
4. \( x : \exists \text{ has-Pet Dog} \)
5. \( x \text{ has-Pet } p \) (2 ;\( \rightarrow \exists \))
6. \( p : \text{Horse} \)
7. \( p : \neg \text{Cat} \) (6 : \( Ax2 \))
8. \( p : \neg \text{Dog} \) (6 : \( Ax3 \))
9. \( x \text{ has-Pet } p \) (3 ;\( \rightarrow \exists \))
10. \( p : \text{Cat} \)
11. \( p : \neg \text{Dog} \) (10 : \( Ax1 \))
12. \( p : \neg \text{Horse} \) (10 : \( Ax2 \))

Contradiction 6,12’

In the same way we find contradictions in the other cases.

**Exercise 5-6  \( ALC \) with Role Hierarchies**

Consider a T-Box with

- the concept definitions and -axioms: \( A \sqsubseteq B \)
- and the following role axiom (Role hierarchy): \( r \sqsubseteq s \) und \( s \sqsubseteq t \)

Check the consistency of the formula \( \exists r \ A \sqcap \forall t \neg B \) with the T-Box using the Tableau Calculus

**Proposed Solution:**

1. \( x : \exists r \ A \)
2. \( x : \forall t \neg B \)
3. \( x \ r \ y \) (1 ;\( \rightarrow \exists \))
4. \( y : A \)
5. \( y : B \) (4 : Axiom \( A \sqsubseteq \neg B \))
6. \( y : \neg B \) (2, 3, \( r \sqsubseteq s, s \sqsubseteq t, \rightarrow \forall \))

Contradiction 5,6
Exercise 5-7  \textit{ALC} mit Role Terms: Blonde Son

Consider the following scenario:

- Daughters are Children.
- Sons are Children.
- Children are either Sons or Daughters.
- Karl's sons are all blonde.
- Karl has a child named Thomas, and Thomas is no daughter.

a) Formulate these facts as \textit{ALC}-formulae using general role axioms. Use the roles \textit{has-Son}, \textit{has-Daughter}, \textit{has-Child} and the single concept \textit{Blonde}.

b) Prove with the Tableau Calculus that Thomas is blonde.

Proposed Solution:

Axioms:

1. $\text{has-Daughter} \sqsubseteq \text{has-Child}$
2. $\text{has-Son} \sqsubseteq \text{has-Child}$
3. $\text{has-Child} \sqsubseteq (\text{has-Son} \sqcup \text{has-Daughter})$
4. $\neg(\text{has-Son} \sqcap \text{has-Daughter})$ (is not actually needed)

Tableau:

6. $\text{Karl} : \forall \text{has-Son Blonde}$
7. $\text{Karl has-Child} \text{ Thomas}$
8. $\text{Karl} \neg \text{has-Daughter} \text{ Thomas}$

9. $\text{Thomas} : \neg \text{Blonde}$ (neg. Theorem)
10. $\text{Karl (has-Son} \sqcup \text{has-Daughter)} \text{ Thomas}$ (7, 3)

11. $\text{Karl has-Son} \text{ Thomas}$
12. $\text{Karl has-Daughter} \text{ Thomas}$ (10 : $\rightarrow_\bot$)

12. $\text{Thomas} : \text{Blonde}$ (6, 11, 2, $\rightarrow_\psi$) Contradiction 8, 11

Contradiction 9, 12
Exercise 5-8  \( \mathcal{ALC} \) with Role Terms: Integration into the Tableau Calculus

Integrate into the Tableau Calculus rules for treating the propositional aspects of role terms.

**Hint:** The modified rule for the universal quantifier \((x : \forall r \psi)\) must do a case distinction:

- Case 1: \(x \neg r y\)
- Case 2: \(x r y \text{ und } y : \psi\)

How can we integrate T-Box axioms for concepts and roles?

**Proposed Solution:**

\[
\begin{array}{llll}
\frac{x s y}{x : \forall r \psi} & \frac{x r \sqcap s y}{x r y} & \frac{x r \sqcup s y}{x r y | x s y} & \frac{x \neg (r \sqcap s) y}{x \neg r y | x \neg s y} \\
\frac{x \neg r y | y : \psi}{x \neg (r \sqcup s) y} & \frac{x r \sqcup s y}{x r y | x s y} & \frac{x \neg (r \sqsubseteq s) y}{x r y y} & \frac{x \neg \neg r y}{x r y}
\end{array}
\]

Contradiction

A T-Box role-axiom \(\varphi\) must be applied to all pairs of variables \(x, y\): add \(x \varphi y\).

Ex.: \(r \sqsubseteq s\) as T-Box axiom generates \(x (r \sqsubseteq s) y\).

One can optimize the method by adding \(x s y\) only if \(r\) is a role name and \(x r y\) is present.